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TORIC GEOMETRY  
IN STRING MODEL BUILDING

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## **Torische Geometrie im Stringmodellbau:**

Wir untersuchen die torische Geometrie als Haupthilfsmittel für den Stringmodellbau in der F-Theorie und in toroidalen Orbifolds der heterotischen Stringtheorie. Für die F-Theorie behandeln wir torische Methoden, mit denen man vierdimensionale elliptische Calabi-Yau Mannigfaltigkeiten als Hyperflächen der torischen Varietäten konstruieren kann. Weiterhin zeigen wir, dass die Branemoduli der F-Theorie einen Teil der komplexen Strukturmoduli der internen Mannigfaltigkeit darstellen, die zur Kompaktifizierung verwendet wird. Für die Untersuchung der Orbifoldmodelle diskutieren wir die Konstruktion der nicht kompakten Calabi-Yau Orbifolds als torische Varietäten und wie man sie auflösen kann. Da die Schnittzahlen der Divisoren wichtig für den Modellbau der Orbifolds sind, argumentieren wir, dass eine vernünftige Schnitttheorie in nicht-kompakten Mannigfaltigkeiten definiert werden kann und geben Regeln vor, mit denen man die Schnittzahlen der aufgelösten nicht-kompakten Orbifolds ausrechnen kann.

## **Toric Geometry in String Model Building:**

We study toric geometry as the main tool for string model building in F-theory and in heterotic toroidal orbifolds. For F-theory we review how elliptic Calabi-Yau fourfolds can be constructed as hypersurfaces of toric varieties. Furthermore, we show that the brane moduli of F-theory are contained in the complex structure moduli of the internal manifold used for the compactification. For studies of orbifold models we discuss how to construct non-compact Calabi-Yau orbifolds as toric varieties and how to resolve them. Since intersection numbers of divisors are important for orbifold model building, we additionally argue that a sensible intersection theory in non-compact manifolds can be defined and give rules to calculate the intersection numbers of resolved non-compact orbifolds.

*A world without string is chaos.*

Lars Smuntz, Quoting his father  
*Mousehunt*

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# Introduction

*Probably the single most prevalent claim advanced by the proponents of a new paradigm is that they can solve the problems that have led the old one to a crisis.*

Thomas S. Kuhn, [Kuh62]

Today, general relativity (GR) and quantum mechanics (QM) or its infinite-dimensional cousin quantum field theory (QFT) are two main streams in fundamental physics. Both theories caused paradigm shifts in gravitational and microscopic physics respectively: from Newton's "simple" gravity law to its geometrization and from the "classical" physics to "quantum" physics and QFT endowed with gauge symmetries describing strong, weak, and electromagnetic interactions. Despite their success, there is no satisfying theory which unites both of them, i.e. all four known interactions. The main topic of today's research is to reconcile GR with QM in quantizing it.

With the advent of string theory another paradigm shift happened for many physicists. Nowadays, string theory is the most promising candidate which could embrace both GR and QM. However, as it is well known, it has many problems. Not being able to naturally reproduce the standard model is one of them.

The aim of this thesis is to develop tools for so-called model building in F-theory and in the heterotic toroidal orbifold context. We proceed as follows: In § 2 we introduce some terminology of algebraic geometry to have the right language for the discussion, especially divisors of a complex manifold. Then we discuss in § 3 elements of toric geometry which will be the main tool for studying and constructing the models. We describe how one can construct compact and non-compact Calabi-Yau manifolds using toric methods. For the study of brane moduli of F-theory, we discuss in § 4 two very important concepts in algebraic geometry, namely sheaves and sheaf cohomology.

After having set the notation and introduced necessary terms, we discuss F-theory and its orientifold limit. One of the main problems of model building is the question how to stabilize the moduli, i.e. complex structure, brane, and Kähler moduli. We show by using sheaf cohomology that the brane moduli which represent an extra class of moduli in type IIB theory besides complex structure and Kähler moduli are a part of complex structure moduli of the fourfold which is used for F-theory compactification. Thus we "only" have to stabilize the complex structure and Kähler moduli. In this thesis, we focus on the complex structure moduli. We briefly discuss one possible mechanism used to stabilize the complex structure moduli.

In the third part of this work we provide toric geometry techniques for the heterotic toroidal orbifold models. We describe the construction of non-compact Calabi-Yau orbifolds as local models of compact toroidal orbifolds. Then we show how to resolve the singularities. To obtain the chiral spectrum in these resolved orbifolds, it is crucial to calculate the intersection numbers of divisors. Here we have a very

peculiar situation. In mathematics the intersection theory of non-compact manifolds is not well defined since the intersection between non-compact cycles is not invariant under deformations. For this reason it has not been considered. We argue that a sensible definition of intersections in non-compact manifold is possible under certain restrictions to deformations and that fractional numbers can occur. We propose a set of rules which can be used to calculate all intersection numbers, including those between non-compact divisors, for various orbifolds. In [GNHT07] it is shown that the number of chiral multiplets of the resolved orbifolds  $\mathbb{C}^2/\mathbb{Z}_3$ ,  $\mathbb{C}^3/\mathbb{Z}_4$ , and  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$  calculated using these rules match those of heterotic orbifolds. This makes us confident that a sensible definition of intersection numbers in non-compact manifolds is possible.

# Algebraic-Geometric Prelude

*Algebra is generous: she often gives more than is asked for.*  
Jean d'Alembert, Quoted in [Mac93]

For further development several concepts of algebraic geometry will be very handy and even necessary. In this chapter, we essentially fix the notation and the language we will be using in later chapters. In order not to bore the reader with too much mathematical definitions and results, we will discuss some concepts in a later chapter.

## 2.1 Basic Definitions

### 2.1.1 Divisors and Line Bundles

Let  $M$  be a complex manifold, not necessarily compact, with complex dimension  $m$  and  $\{U_\alpha\}$  an open cover of  $M$ . We mainly follow the treatment of [GH78, § 1.1].

An **analytic hypersurface**  $V$  of  $M$  is a subset such that for every  $x \in V$  there exists an open set  $U_x \subset M$  whose intersection with  $V$  is the zero locus of a holomorphic function  $f_x$ . Every holomorphic function whose zero locus contains  $U_x \cap V$  is divisible by  $f_x$ . This function  $f_x$  is called the **local defining function** for  $V$  near  $x$ . Consider the Jacobian of  $f_x$

$$\mathcal{J}(f_x)(z) = (\partial_{z_1} f_x(z), \dots, \partial_{z_m} f_x(z)),$$

where  $z = (z_1, \dots, z_m)$  is the local coordinate system in  $U_x$ . We call a point  $y \in U_x \cap V$  singular if

$$\mathcal{J}(f_x)(y) = (0, \dots, 0).$$

If  $V$  does not have any singular point,  $V$  is a submanifold of codimension 1. The hypersurface  $V$  is called **irreducible** if it cannot be written as the union of two smaller analytic hypersurfaces.

Let  $x \in V$  and  $f_x$  be a local defining function for  $V$  near  $x$ . Let  $g$  be a holomorphic function defined near  $x$ . We define the **order** of  $g$  to be the largest integer  $a$  denoted by  $\text{ord}_{V,x} g$  such that

$$h = \frac{g}{f_x^a}$$

is holomorphic around  $x$ . The order of  $g$  is well defined and is always a non-negative integer. It does not depend on  $x$  and is denoted by  $\text{ord}_V(g)$ . If  $g$  does not vanish on  $V$ , then  $\text{ord}_V(g) = 0$ . The order  $\text{ord}_V(g)$  tells us how many times  $g$  contains the local defining function of  $V$ .

For  $g$  and  $h$  any holomorphic functions we have

$$\text{ord}_V(g \cdot h) = \text{ord}_V(g) + \text{ord}_V(h).$$

Let  $f$  be a meromorphic function on  $M$ . It can be written locally as

$$f = \frac{g}{h}$$

with  $g$  and  $h$  holomorphic and relatively prime<sup>1</sup>. We define

$$\text{ord}_V(f) = \text{ord}_V(g) - \text{ord}_V(h).$$

A **divisor**  $D$  of  $M$  is a finite formal linear sum

$$D = \sum_i a_i \cdot V_i, \quad a_i \in \mathbb{Z}$$

of irreducible analytic hypersurfaces  $V_i$  of  $M$ . We write  $\text{Div}(M)$  for the set of all divisors of  $M$ . A divisor is called **effective** if  $a_i \geq 0$  for all  $i$ . Let  $f$  be a meromorphic function on  $M$  defined locally as  $g/h$  and not identically zero. We associate the divisor  $\text{div}(f)$  to  $f$  by setting

$$\text{div}(f) = \sum_V \text{ord}_V(f) \cdot V = \text{div}(f)_0 - \text{div}(f)_\infty,$$

where

$$\text{div}(f)_0 = \sum_V \text{ord}_V(g) \cdot V \quad \text{and} \quad \text{div}(f)_\infty = \sum_V \text{ord}_V(h) \cdot V.$$

The sum runs over all irreducible hypersurfaces of  $M$ . However the sum is finite since the order of  $h$  or  $g$  for a hypersurface is zero if the zero locus of  $h$  or  $g$  do not contain the hypersurface. We call a divisor of form  $D = \text{div}(f)$  a **principal divisor**.

Divisors can be defined in a different way. Let  $\mathcal{M}(M)$  be the set of meromorphic functions and  $\mathcal{O}(M)$  the set of holomorphic functions on  $M$ . Let  $\mathcal{M}^*(U)$  and  $\mathcal{O}^*(U)$  be the sets of meromorphic functions not identically zero and nonzero, i.e. nowhere zero, holomorphic functions on an open set  $U \subset M$ . Let  $\{U_\alpha\}$  be an open cover of  $M$  as before. A divisor  $D$  is given by

$$f_\alpha \in \mathcal{M}^*(U_\alpha)$$

for each  $U_\alpha$  with

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$$

through

$$D = \sum_V \text{ord}_V(f_\alpha) \cdot V,$$

where for each  $V$  we choose  $\alpha$  such that  $V \cap U_\alpha \neq \emptyset$ . The functions  $f_\alpha$  are called the **local defining functions** of  $D$ . Let us unwind the definition. For each  $U_\alpha$  we pick a meromorphic function  $f_\alpha$ . We demand that  $f_\alpha$  and  $f_\beta$  coincide on the intersection  $U_\alpha \cap U_\beta$  up to a nonzero holomorphic function, i.e. they have the same zeros and poles on  $U_\alpha \cap U_\beta$ . We sum over all hypersurfaces  $V$  of  $M$  which have non-trivial intersection with  $U_\alpha$ . The order  $\text{ord}_V(f_\alpha)$  is the multiplicity  $V$  is summed with. This

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<sup>1</sup>The notion prime in this context is naturally understandable. It means that  $f$  and  $g$  do not have any common factor  $\tilde{h}$  which is also a holomorphic function.

is well-defined even if  $V$  has non-trivial intersection with two or more open sets since the local defining functions have the same zeros and poles in the intersection. Thus it does not matter which  $f_\alpha$  we take.

If  $M$  is not smooth, but contains singularities, i.e.  $M$  is a **variety**<sup>2</sup>, both definitions in general do not yield the same set of divisors<sup>3</sup>. The first definition describes the set of the **Weil divisors** and the second definition the set of the **Cartier divisors**. Every Cartier divisor is a Weil divisor, but not vice versa.

We say two divisors  $D$  and  $D'$  are **linearly equivalent** if

$$D - D' = \text{div}(f)$$

for a  $f \in \mathcal{M}^*(M)$ . We write

$$D \sim D'.$$

Clearly  $\sim$  is an equivalence relation.

If we have a bundle  $E$  of any kind, e.g. vector bundles, fiber bundles, etc., we will use following notation:

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

The fiber is denoted by  $F$ , e.g. vector spaces  $\mathbb{R}^n$  and  $\mathbb{C}^n$  for vector bundles, and the base by  $B$ .

Let  $\{U_\alpha\}$  be an open cover of  $M$  and

$$g_{\alpha\beta} : U_\alpha \longrightarrow U_\beta \quad , \quad g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$$

the transition functions of a line bundle  $L$  on  $M$ . A **holomorphic section** of  $L$  over  $U \subset M$  is given by a collection of holomorphic functions  $s_\alpha$  on  $U \cap U_\alpha$  with

$$s_\alpha = g_{\alpha\beta} \cdot s_\beta \quad \text{in} \quad U \cap U_\alpha \cap U_\beta.$$

If  $U = M$ , then  $s$  is a **global holomorphic section**. A **meromorphic section** is defined analogously with  $s_\alpha$  being now meromorphic functions on  $U$ . The quotient of two meromorphic sections is a well-defined meromorphic section.

The first Chern class of a line bundle  $L$  denoted by  $c_1(L)$  is a characteristic quantity for  $L$ . The first Chern class is an element of  $H^2(M, \mathbb{Z})$ , i.e.  $c_1(L) \in H_{\text{dR}}^2(M)$  with the additional property that the integrals of  $c_1(L)$  over complex one-dimensional submanifolds are  $\mathbb{Z}$ -valued, where  $H_{\text{dR}}^2(M)$  denotes the second de Rham cohomology group of  $M$ . Additionally, one can show that  $c_1(L) \in H^{1,1}(M)$  [GH78, p.416]. A line bundle  $L$  on  $M$  is determined up to isomorphism by its first Chern class. If two line bundles are isomorphic, then their first Chern classes are equal. The group of all line bundles on  $M$  up to isomorphism is called the **Picard group** and is denoted by  $\text{Pic}(M)$ . The group operation is given by the tensor product<sup>4</sup>  $\otimes$  and the inverse of a line bundle  $L$  is its dual line bundle  $L^*$ . By the group operation the transition functions get multiplied or divided respectively. The unit of this group is given by the trivial line bundle.

<sup>2</sup>A variety is a manifold with possible singularities, i.e. variety with no singularities is a manifold.

<sup>3</sup>For more details see for example [Sha94, p.256].

<sup>4</sup>Some authors write the tensor product additively, e.g. instead of  $L \otimes L \cong L^2$  it is written  $2L$ .

There is a nice correspondence<sup>5</sup> between divisors and line bundles on  $M$ . Let  $D$  be given by  $\{f_\alpha\}$  with  $f_\alpha \in \mathcal{M}^*(U_\alpha)$  and

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta), \quad (2.1)$$

i.e.  $g_{\alpha\beta}$  is a non-zero holomorphic function in  $U_\alpha \cap U_\beta$ . We have furthermore the cocycle condition

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{for } U_\alpha \cap U_\beta \cap U_\gamma.$$

Thus we have a line bundle  $L$  with transition functions  $\{g_{\alpha\beta}\}$ . This line bundle is called the **associated line bundle** of  $D$  and is denoted<sup>6</sup> by  $[D]$ .

The correspondence

$$[\ ] : \text{Div}(M) \longrightarrow \text{Pic}(M)$$

is a group homomorphism, i.e.

$$[D + D'] = [D] \otimes [D'] \quad \text{and} \quad [-D] = [D]^{-1}.$$

The homomorphism  $[\ ]$  descends to the quotient

$$[\ ] : \text{Div}(M)/\cong \longrightarrow \text{Pic}(M),$$

where  $\sim$  denotes the linear equivalence of divisors. This homomorphism is *injective*. The line bundle  $[D]$  is trivial iff  $D = \text{div}(f)$ . For **projective** manifolds it is an *isomorphism* [GH78, p.161], where projective means that the manifold can be holomorphically embedded in  $\mathbb{C}P^n$  for some  $n$ . The reason is that every line bundle  $L$  of a projective manifold has a global meromorphic section  $s$ . Thus we define the inverse mapping through

$$\text{div}(s) = \sum_V \text{ord}_V(s) \cdot V.$$

Note that if  $s$  is holomorphic,  $\text{div}(s)$  is effective.

The first Chern class of the associated line bundle of a divisor  $D$ , i.e.  $c_1([D])$ , represents the Poincaré dual of the homology cycle of  $D$ . We will therefore write  $D$  for  $c_1([D])$ . One important consequence of this fact is that the principal divisors are homologous to zero, i.e. the corresponding Chern class is zero.

## 2.1.2 Canonical Bundle

Let  $M$  be a complex manifold with complex dimension  $m$ . Any complex manifold is a real differentiable manifold and we can look at its **tangent bundle**  $T_{\mathbb{R}}M$  which is a *real*  $2m$ -dimensional vector bundle on  $M$ . The **complexified tangent bundle** of  $M$  denoted by

$$T_{\mathbb{C}}M := T_{\mathbb{R}}M \otimes \mathbb{C}$$

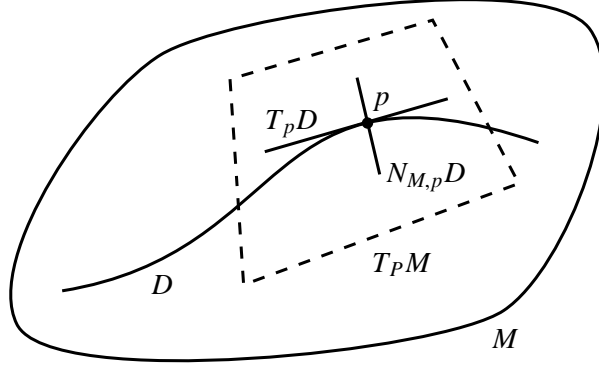
has a natural decomposition in two pieces

$$T_{\mathbb{C}}M = TM \oplus \overline{TM},$$

where we write  $TM$  for the **holomorphic tangent bundle** and  $\overline{TM}$  for the **anti-holomorphic tangent bundle**. It is clear that  $T_{\mathbb{C}}M$  is a *complex*  $2m$ -dimensional vector bundle on  $M$ . Let  $T^*M$  and  $\overline{T^*M}$

<sup>5</sup>Strictly speaking, this correspondence is valid only for Cartier divisors. In § 3.2.2 this fact will play a role.

<sup>6</sup>It is often denoted by  $\mathcal{O}_M(D)$ .



**Figure 2.1:** The normal bundle of a divisor  $D$  in  $M$  at the point  $p \in M$

denote the dual bundle of  $TM$  and  $\overline{TM}$  respectively. The **canonical bundle**  $KM$  is the determinant bundle of  $T^*M$ , i.e.

$$KM = \det T^*M = \bigwedge^m T^*M.$$

It is clear that  $KM$  is a line bundle over  $M$ . The sections on  $KM$  are the  $(m, 0)$ -forms of  $M$ . The dual bundle of  $KM$  denoted by  $KM^{-1}$  is called for obvious reason the **anticanonical bundle**. As it is well known  $KM$  and  $KM^{-1}$  play an essential role for the discussion of Calabi-Yau manifolds. We write  $K_M$  for the corresponding divisor to  $KM$ , the **canonical divisor**. Obviously the **anticanonical divisor** is  $-K_M$ .

### 2.1.3 Adjunction Formulae

We look at the canonical bundle of a divisor  $D$  in  $M$ . It will be important because many Calabi-Yau manifolds will be constructed as a divisor of a rather simple ambient manifold.

In order to determine  $KD$  we need the concept of the **normal bundle** of a divisor<sup>7</sup>. The normal bundle  $N_M D$  is defined through the following short exact sequence

$$0 \longrightarrow TD \xrightarrow{i} TM|_D \xrightarrow{p} N_M D \longrightarrow 0. \quad (2.2)$$

Since the sequence is exact,  $i$  is injective and  $p$  is surjective, i.e. the maps  $i$  and  $p$  are inclusion and projection respectively. At each point  $p \in M$  the vector space  $T_p M$  is spanned by the basis vectors of  $T_p D$  and  $N_{M,p} D$

$$T_p D \oplus N_{M,p} D = T_p M \quad \forall p \in M,$$

where the subscript  $p$  means the bundle at the point  $p$ . Figure 2.1 illustrates the concept of the normal bundle.

The **first adjunction formula** tells us that

$$N_M D = [D]|_D, \quad (2.3)$$

<sup>7</sup>It is clear that the normal bundle can be defined for any subvariety of  $M$ . We restrict ourselves to divisors because we do not need this concept for subvarieties with codimension greater than 1.

where  $[D]|_D$  means the usual restriction of  $[D]$  to  $D$ . Since a divisor is a codimension one object, its normal bundle is complex one-dimensional, i.e. a line bundle.

The **second adjunction formula** allows us to determine  $KD$

$$KD = KM|_D \otimes N_M D = (KM \otimes [D])|_D, \quad (2.4)$$

where in the second equality we used the first adjunction formula. We will use these formulae very often as we go along.

### 2.1.4 Properties of the Chern Class

We list general properties of the Chern Class as we will use it extensively. One can find most of these properties with corresponding proofs in [GH78, p.407].

Let  $c(E)$  denote the total Chern class of a complex vector bundle  $E$ . We denote the  $i$ -th Chern class by  $c_i(E)$ . Furthermore we write  $c(M)$  for  $c(TM)$ . The  $i$ -th Chern class of  $TM$  is an element of  $H^{2i}(M, \mathbb{Z})$ , the  $2i$ -th cohomology class whose elements give integer numbers when integrated over  $2i$ -cycles of  $M$ . Additionally, one can show that  $c_i(M) \in H^{i,i}(M)$  which is the Dolbeault cohomology group of degree  $(i, i)$ .

One of the most important properties is the **naturalness**. Let  $M$  and  $N$  be complex manifolds and  $f$  a  $\mathcal{C}^\infty$ -map

$$f : M \longrightarrow N.$$

Let furthermore  $E$  be a holomorphic vector bundle on  $N$  and  $f^*E$  be the pulled-back vector bundle<sup>8</sup> on  $M$ :

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Then

$$c(f^*E) = f^*c(E). \quad (2.5)$$

The Chern class behaves *natural* under the pull-back in the sense that the Chern class of the pulled-back bundle is equal to the pull-back of the Chern class.

Let  $V', V''$  and  $V$  be three holomorphic vector bundles on  $M$ . Assuming we have a short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

we obtain for the total Chern class of  $V$

$$c(V) = c(V') \cdot c(V'').$$

This includes the case  $V = V' \oplus V''$ . This formula is called the **Whitney product formula**. See for example [Har77, p.430] or [BT82, Rem. 21.6].

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<sup>8</sup>For the definition of the pull-back of a vector bundle see for example [BT82, p.56].

Let  $E$  be a holomorphic vector bundle of rank  $k$ , then

$$c_1(E \otimes L) = c_1(E) + k \cdot c_1(L).$$

Thus if  $L$  and  $L'$  are complex line bundles on a complex manifold  $M$ , then

$$c_1(L \otimes L') = c_1(L) + c_1(L'). \quad (2.6)$$

Let  $E$  be a holomorphic vector bundle on a complex manifold  $M$  and let  $E^*$  denote the dual bundle of  $E$ , then

$$c_k(E^*) = (-1)^k \cdot c_k(E). \quad (2.7)$$

The canonical bundle  $KM$  has the first Chern class

$$c_1(KM) = -c_1(M). \quad (2.8)$$

## 2.2 Complex Projective Spaces

One could say that complex projective spaces  $\mathbb{C}P^n$  are the most important complex manifolds. For convenience we write  $\mathbb{P}^n$  for  $\mathbb{C}P^n$ . They are perfectly suited to illustrate the ideas of previous sections without being trivial. We first fix the notation.

### 2.2.1 Notations for $\mathbb{P}^n$

For  $\mathbb{P}^n$  there is a very natural line bundle, called the **tautological bundle**<sup>9</sup> denoted by  $\mathcal{O}_{\mathbb{P}^n}(-1)$ . It is defined as follows [Huy04, p.69]

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in \ell\},$$

i.e. for every point in  $\mathbb{P}^n$  we take the line defined by the point as its fiber<sup>10</sup>. If we cover  $\mathbb{P}^n$  with the usual open sets defined by

$$U_i = \{z_i \neq 0\} \cap \mathbb{P}^n,$$

then the transition functions of this line bundle are

$$\varphi_{ij} = \frac{z_i}{z_j} \quad \text{for } U_i \cap U_j.$$

Let  $\tilde{\omega} \in H^2(\mathbb{P}^n, \mathbb{Z})$  denote the first Chern class of  $\mathcal{O}_{\mathbb{P}^n}(-1)$ .

We call the dual line bundle, i.e. the inverse line bundle, of  $\mathcal{O}_{\mathbb{P}^n}(-1)$  the **hyperplane bundle**. It is then natural to write  $\mathcal{O}_{\mathbb{P}^n}(1)$  for it. We will soon see that the hyperplane bundle is the associated line bundle for every hyperplane  $H$  in  $\mathbb{P}^n$  which is defined as the zero locus of a homogeneous polynomial of degree 1. The transition functions are

$$\psi_{ij} = \frac{z_j}{z_i} \quad \text{for } U_i \cap U_j.$$

Let  $\omega \in H^2(\mathbb{P}^n, \mathbb{Z})$  denote the first Chern class of  $\mathcal{O}_{\mathbb{P}^n}(1)$ . It is clear using (2.7) that  $\omega = -\tilde{\omega}$ .

---

<sup>9</sup>Note that in the literature it is sometimes called the **canonical line bundle** of  $\mathbb{P}^n$ . However, the word *canonical* is overloaded and one can confuse it with the canonical bundle  $K\mathbb{P}^n$  which also is a line bundle.

<sup>10</sup>It's so tautological!

	$V_0$	$V_1$
$U_0$	1	$z_1/z_0$
$U_1$	$z_0/z_1$	1
$U_2$	$z_0/z_2$	$z_1/z_2$
$U_3$	$z_0/z_3$	$z_1/z_3$

**Table 2.1:** Local defining functions for hyperplanes  $V_0$  and  $V_1$

The total Chern class of  $\mathbb{P}^n$  is [GH78, p.409]

$$c(\mathbb{P}^n) = (1 + \omega)^{n+1}.$$

The  $i$ -th Chern classes can be obtained by expanding  $c(\mathbb{P}^n)$ , e.g. the first Chern class is

$$c_1(\mathbb{P}^n) = (n + 1)\omega.$$

Using (2.8), we obtain

$$c_1(K\mathbb{P}^n) = -(n + 1)\omega.$$

If we write  $\mathcal{O}_{\mathbb{P}^n}(m)$  for  $\mathcal{O}_{\mathbb{P}^n}(1)^m$  and  $\mathcal{O}_{\mathbb{P}^n}(-m) = \mathcal{O}_{\mathbb{P}^n}(-1)^m$ , the line bundle  $\mathcal{O}_{\mathbb{P}^n}(-(n + 1))$  has the same first Chern class as  $K\mathbb{P}^n$  employing (2.6). Since a line bundle is determined by the first Chern class, we obtain

$$K\mathbb{P}^n = \mathcal{O}_{\mathbb{P}^n}(-(n + 1)).$$

The Picard group of  $\mathbb{P}^n$  is particularly simple [GH78, p.145]

$$\text{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z},$$

i.e. every line bundle on  $\mathbb{P}^n$  is a power of  $\mathcal{O}_{\mathbb{P}^n}(1)$ . Now it is clear why the notation  $\mathcal{O}_{\mathbb{P}^n}(m)$  makes sense. Additionally, global sections of  $\mathcal{O}_{\mathbb{P}^n}(m)$  correspond to homogeneous polynomials of degree  $m$ .

## 2.2.2 Divisors in $\mathbb{P}^n$

Let us consider two hyperplanes  $V_0$  and  $V_1$  of  $\mathbb{P}^3$  with homogeneous coordinate  $z = (z_0, z_1, z_2, z_3)$ . The first hyperplane  $V_0$  is defined as the zero locus of the homogeneous polynomial  $f_0(z) = z_0$ , i.e.

$$V_0 = \{0, z_1, z_2, z_3\} \cap \mathbb{P}^3 \cong \mathbb{P}^2.$$

The second hyperplane  $V_1$  is defined analogously with  $f_1(z) = z_1$ . The hyperplane  $V_1$  is also isomorphic to  $\mathbb{P}^2$ .

To analyze the hyperplanes as divisors of  $\mathbb{P}^3$ , we first cover  $\mathbb{P}^3$  with the usual open sets

$$\begin{aligned} U_0 &= \{z_0 \neq 0\} \cap \mathbb{P}^3, \\ U_1 &= \{z_1 \neq 0\} \cap \mathbb{P}^3, \\ U_2 &= \{z_2 \neq 0\} \cap \mathbb{P}^3, \\ U_3 &= \{z_3 \neq 0\} \cap \mathbb{P}^3. \end{aligned}$$

Table 2.1 shows the local defining functions  $g_i$  and  $h_i$  for  $V_0$  and  $V_1$  respectively. Functions on  $U_i$  should be well defined, i.e. invariant under the  $\mathbb{C}^*$ -action since  $U_i$  are not projective. This is the reason why the local defining functions  $g_i$  and  $f_i$  are of the given form. Since the coordinate  $z_i$  is non-zero in  $U_i$ , we

divide  $f_i$  by it to obtain an invariant function. The constant function 1 in  $U_0$  for  $V_0$  also makes sense since  $V_0 \cap U_0 = \emptyset$ . Additionally, we easily see that

$$\frac{g_i}{g_j}, \frac{h_i}{h_j} \in \mathcal{O}^*(U_i \cap U_j).$$

Thus we have a divisor description for  $V_0$  and  $V_1$ .

We now construct the associated line bundles  $[V_0]$  and  $[V_1]$ . The transition functions are for both divisors using (2.1)

$$\varphi_{ij} = z_j/z_i \quad \text{for } U_i \cap U_j,$$

i.e.  $[V_0]$  and  $[V_1]$  are identical. This means

$$V_0 \sim V_1$$

since the homomorphism  $[\cdot]$  is injective. This we can see also at the level of local defining functions of  $V_0$  and  $V_1$ . From the Table 2.1 we see again that

$$h_i = g_i \cdot \frac{z_1}{z_0} \quad \forall i.$$

Thus we obtain

$$V_1 = V_0 + \text{div}\left(\frac{z_1}{z_0}\right).$$

Since  $\text{div}(f)$  with a global meromorphic section  $f$  defines a homologous trivial divisor,  $V_0$  and  $V_1$  are represented by the same class in the homology.

Obviously we can generalize this construction for  $\mathbb{P}^n$  with more general homogeneous polynomials  $f_0$  and  $f_1$  both of degree  $d$  whose zero loci define two hypersurfaces  $D_0$  and  $D_1$ . The local defining functions are

$$g_i = \frac{f_0}{z_i^d} \quad \text{and} \quad h_i = \frac{f_1}{z_i^d}.$$

We see again that

$$h_i = g_i \cdot \frac{f_1}{f_0}.$$

Both divisors are linearly equivalent because

$$D_1 = D_0 + \text{div}\left(\frac{f_1}{f_0}\right).$$

Moreover, we have

$$D_0 \sim D_1 = dV_0 + \text{div}\left(\frac{f_1}{z_0^d}\right),$$

where the local defining functions for  $dV_0$  are  $z_0^d/z_i^d$ . The associated line bundle is obviously  $[V_0]^d$  for  $D_0, D_1$  and  $dV_0$ .

### 2.2.3 Calabi-Yau Hypersurfaces in $\mathbb{P}^n$

In this section we show for example that the quintic in  $\mathbb{P}^4$  is Calabi-Yau. The total Chern class of  $\mathbb{P}^4$  is

$$c(\mathbb{P}^4) = (1 + \omega)^5.$$

Let  $V$  be a hypersurface defined as the zero locus of a homogeneous polynomial of degree  $d$  in  $\mathbb{P}^4$ . We know that  $V$  is linearly equivalent to  $dV_0$ . We apply the first and the second adjunction formulae (2.3,2.4) to obtain

$$KV = \left( K\mathbb{P}^4 \otimes [V_0]^d \right) \Big|_V. \quad (2.9)$$

We obtain the first Chern class of  $V$  by applying<sup>11</sup>  $c_1(\cdot)$ ,

$$\begin{aligned} -c_1(V) &= c_1(KV) = c_1(K\mathbb{P}^4) + c_1([V_0]^d) \\ &= -c_1(\mathbb{P}^4) + d \cdot \omega \\ &= (-5 + d)\omega. \end{aligned}$$

We want to have vanishing first Chern class for  $V$ . Thus we need  $d = 5$ , i.e. a quintic. We can trivially generalize this analysis to  $\mathbb{P}^n$ , essentially just by replacing 4 by  $n$ . The zero locus of a homogeneous polynomial of degree  $(n + 1)$  in  $\mathbb{P}^n$  is Calabi-Yau.

---

<sup>11</sup>We can ignore the restriction to  $V$  when we apply  $c_1(\cdot)$  to (2.9) due to the naturalness of the Chern class. The restriction is the pull-back of the inclusion and the Chern class behaves naturally under the pull-back.

# Elements of Toric Geometry

*There are surely worse things than being wrong,  
and being dull and pedantic are surely among them.*

Mark Kac, Quoted in [Mac93]

Toric geometry is a very fruitful and exciting topic, mathematically and physically. Mathematically it is a beautiful playground for deep algebro-geometrical theorems. Why it is also relevant for physics or at least for string theory, we will describe in this chapter. Following Kac's remark we will take a pragmatic approach and will not be mathematically rigorous and rely heavily on pictures and reader's intuition.

## 3.1 Basics of Toric Geometry

First of all, toric geometry *cannot* describe the conventional torus,  $\mathbb{C}/\Lambda$ , where  $\Lambda$  is a lattice. The torus in toric geometry is the **algebraic torus**,  $(\mathbb{C}^*)^n$ . A toric variety of complex dimension  $n$  is a variety which contains the algebraic torus  $(\mathbb{C}^*)^n$  as an open subset and on which an action of this torus is defined. In this section we describe how toric varieties are constructed. The reader should remind himself that this is only a *recipe* and therefore we set no great store by mathematical rigour or background<sup>1</sup>.

### 3.1.1 Fans

The very basic object of a toric variety is called fan. To be able to define a fan, we need some more terminology. For our discussion a **lattice**  $N$  will be  $\mathbb{Z}^n$  as shown in Figure 3.1(a). Let  $M \cong \mathbb{Z}^n$  denote the **dual lattice** to  $N$ . We will need  $M$  when we discuss Calabi-Yau hypersurfaces in toric varieties.

Let  $S = \{v_i\}$  be a finite set of vectors in  $N_{\mathbb{R}}$ , where<sup>2</sup>  $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^n$ . We now define the **convex polyhedral cone**  $\sigma$

$$\sigma = \text{Cone}(S) = \{\sum_i \lambda_i v_i \mid \lambda_i \geq 0\}.$$

A **strongly convex polyhedral cone** is a convex polyhedral cone which satisfies

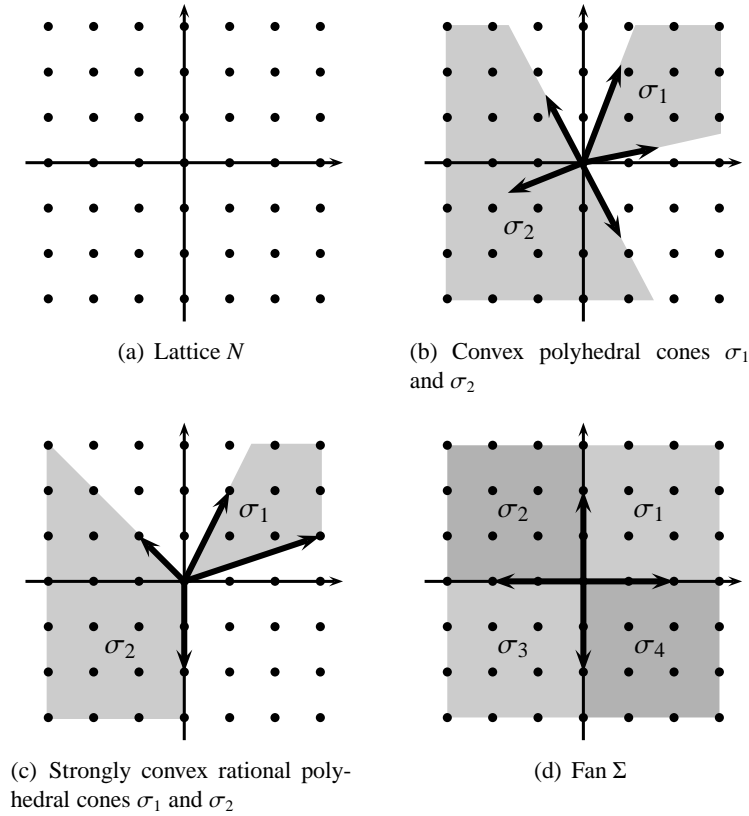
$$\sigma \cap (-\sigma) = \emptyset,$$

where  $(-\sigma) = \text{Cone}(-S)$ . Strong cones thus do not contain a line through the origin. For example, the cone  $\sigma_1$  in Figure 3.1(b) is an example for a strongly convex cone. The cone  $\sigma_2$  in contrast is only convex.

---

<sup>1</sup>In § A.1 we describe the construction in more detail.

<sup>2</sup>To be more precise, we should write  $N \otimes_{\mathbb{Z}} \mathbb{R}$ . For our discussion however, this is not important.



**Figure 3.1:** Basic objects of toric geometry

Because three attributes are not enough, we add one more and look at the **strongly convex rational polyhedral cone**. We demand that the vectors  $v_i$  in  $S$  are not just elements of  $N_{\mathbb{R}}$ , but also of  $N$ . Figure 3.1(c) shows examples of two-dimensional strongly convex rational polyhedral cones.

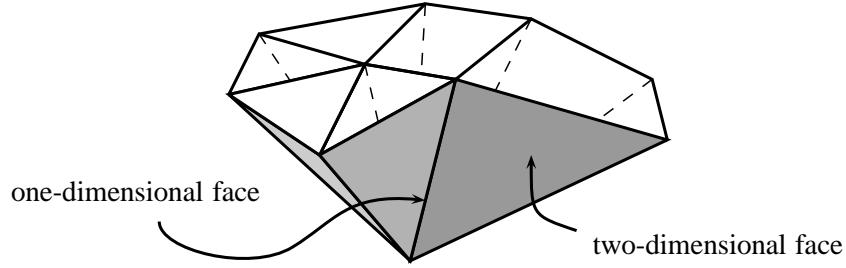
We use Figure 3.2 to define a **face**  $\tau$  of a cone  $\sigma$  since the concept is intuitively clear. A face of codimension 1 is called a **facet**.

Now we define the **fan**  $\Sigma$ . It is a collection of cones satisfying following three properties:

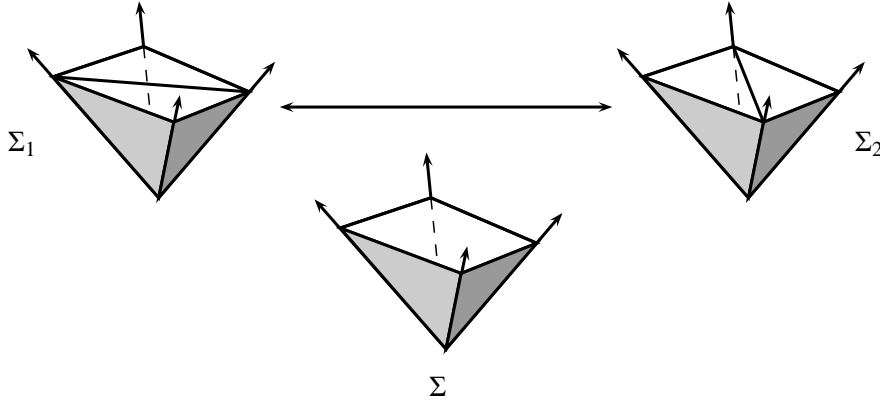
1. Every cone in  $\Sigma$  is a strongly convex rational polyhedral cone.
2. If  $\sigma \in \Sigma$  and  $\tau$  is a face of  $\sigma$ , then  $\tau \in \Sigma$ .
3. If  $\sigma_1, \sigma_2 \in \Sigma$  and  $\sigma_1 \neq \sigma_2$ , then  $\sigma_1 \cap \sigma_2$  is a face of each.

Since we are going to work only with strongly convex rational polyhedral cones, we just write **cone** from now on. The third point in the definition of the fan means that two cones of equal dimension are not allowed to have a non-trivial intersection which is not a face of each. This means for example that a cone is not contained in any other equally dimensional cones. It should be noted that every fan contains the zero-dimensional cone which is the origin. Figure 3.1(d) shows a fan with four two-dimensional cones  $\sigma_1, \sigma_2, \sigma_3$ , and  $\sigma_4$ . It contains also the zero-dimensional cone and four one-dimensional cones generated by the four vectors shown. We use the notion **generators** of the one-dimensional cones and the vectors interchangeably<sup>3</sup>. See also Figure 3.2 for a three-dimensional example of a fan.

<sup>3</sup>We only consider fans which contains all one-dimensional cones generated by the vectors in the fan.



**Figure 3.2:** Example of a fan in  $N_{\mathbb{R}} \cong \mathbb{R}^3$



**Figure 3.3:** Possible choices of cones in a fan

It should be noted that it is not enough to just give the one-dimensional generators to specify a fan. See the fans  $\Sigma$ ,  $\Sigma_1$  and  $\Sigma_2$  in Figure 3.3. They have the same four one-dimensional generators, but have different cones in them. Thus we also have to specify which cones the fan contains. The transition between  $\Sigma_1$  and  $\Sigma_2$  is called a **flop**. Examples of flops will be discussed in § 6.3.

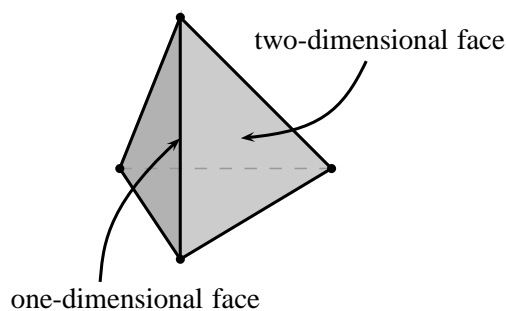
There is another important object in toric geometry called **rational polyhedron**<sup>4</sup>  $P$ . Let  $S$  be a finite set of points in the lattice  $N$ . We call these points **vertices**. *Rational* means that all vertices are not only points of  $N_{\mathbb{R}}$ , but also of  $N$ . From now on we write **polyhedron** for rational polyhedron. The polyhedron  $P \in N_{\mathbb{R}}$  is the convex hull of points in  $S$

$$P = \left\{ \sum_{i=1}^n \lambda_i p_i \mid p_i \in S \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}.$$

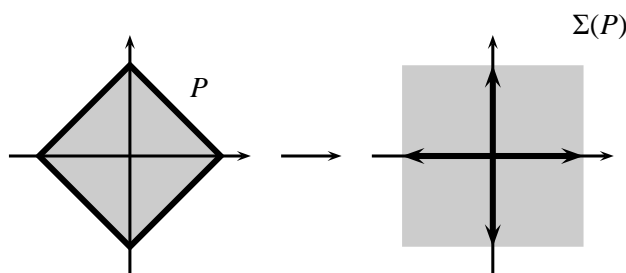
**Faces** of  $P$  are obviously defined as in the case of cones. See Figure 3.4 for examples. Faces of codimension 1 are called again **facets**. We can associate a fan  $\Sigma(P)$  to  $P$ . The cones in  $\Sigma(P)$  are the cones over the faces of  $P$ . See Figure 3.5 for a simple example. It is clear that vertices form the generators in  $\Sigma(P)$ . We assume that every polyhedron contains the origin. If we say we construct a toric variety from a polyhedron, we mean that we do it through the associated fan<sup>5</sup>. Note that toric varieties constructed from a polyhedron are projective [Cox05, Thm. 4.1]. For an example of non-projective toric variety, consider a two-dimensional fan which consists of two one-dimensional cones for which the origin is the

<sup>4</sup>In the literature **polytopes** are often used instead of polyhedrons. Polytopes are bounded polyhedrons. For our purposes they are the same because we only consider bounded polyhedrons.

<sup>5</sup>One normally starts with a polyhedron  $P^*$  in the dual lattice  $M_{\mathbb{R}}$ . From  $P^*$  one determines the so-called **normal fan** which is in  $N_{\mathbb{R}}$ . To rigorously define it would take much more space and care. We skip it [Ful93, p.26].



**Figure 3.4:** Faces of a polyhedron



**Figure 3.5:** Associated fan of a polyhedron

only face they share. This fan does not stem from a polyhedron. Thus the toric variety constructed from this polyhedron will not be projective.

### 3.1.2 Homogeneous Coordinates Construction

There are many ways to construct a toric variety from a fan. To name the most typical two, there are the construction using  $\text{Spec}(\cdot)$ <sup>6</sup> and the homogeneous coordinate construction. The first construction is more classical and somewhat *local*. To read more about this method, see for example [Oda85] or [Ful93]. The homogeneous coordinates method is easier and is a *global* approach, see [Cox05, Lect. 3] or [HKK<sup>+</sup>03, § 7] and for a proof [Cox93, Thm. 2.1]. We will use the second method. By doing so, it is easy to see that toric varieties are *generalized* weighted projective spaces. Therefore we first review the construction of  $\mathbb{P}^n$  out of  $\mathbb{C}^{n+1}$  and redo the steps for toric varieties.

To obtain  $\mathbb{P}^n$ , we do the following:

**Step 1:** Start with  $\mathbb{C}^{n+1}$ .

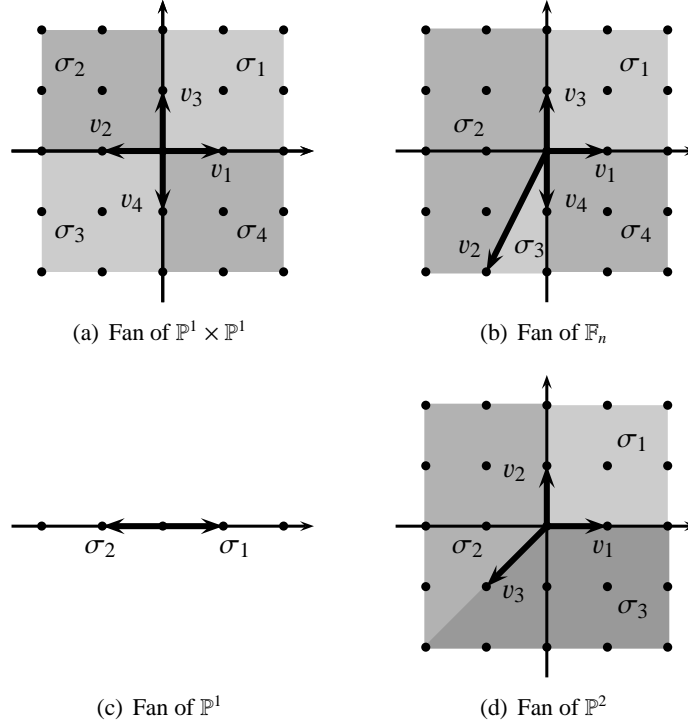
**Step 2:** Subtract  $\{0\}$  from  $\mathbb{C}^{n+1}$ .

**Step 3:** Define an action of  $\mathbb{C}^*$  on  $\mathbb{C}^{n+1} - \{0\}$ :  $(z_1, \dots, z_{n+1}) \rightarrow \lambda(z_1, \dots, z_{n+1})$ .

**Step 4:** Take the quotient of  $\mathbb{C}^{n+1} - \{0\}$  by the  $\mathbb{C}^*$ -action.

We go through the steps with an easy example. We take Figure 3.6(a) as the fan for this example. Since the lattice  $N$  is two-dimensional, we will get a complex two-dimensional toric variety. In general, if we want to construct complex  $n$ -dimensional toric varieties,  $N = \mathbb{Z}^n$ .

<sup>6</sup> $\text{Spec}(R)$  means the set of all prime ideals of a ring  $R$ .



**Figure 3.6:** Fans of simple toric varieties

**Step 1:** Let  $(V)$  denote the set of generators of one-dimensional cones

$$(V) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}. \quad (3.1)$$

We have four generators, so we start with  $\mathbb{C}^4$ . We assign to each  $v_i$  a coordinate  $z_i$  of  $\mathbb{C}^4$  which is called a homogeneous coordinate.

**Step 2:** We now have to subtract a set of points from  $\mathbb{C}^4$ . This set is called the **exceptional set**<sup>7</sup>  $Z(\Sigma)$ . To determine  $Z(\Sigma)$ , we have to determine all sets of  $v_i$  which do not span a cone<sup>8</sup> in  $\Sigma$ . In this example these are

$$\{v_1, v_2\}, \{v_3, v_4\}, \{v_i, v_j, v_k\} \text{ for } i, j, k \text{ all different, and } \{v_1, v_2, v_3, v_4\}.$$

We set

$$\begin{aligned} Z(\Sigma) &= \{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\} \cup \{z_1 = z_2 = z_3 = 0\} \cup \{z_1 = z_2 = z_4 = 0\} \\ &\quad \cup \{z_1 = z_3 = z_4 = 0\} \cup \{z_2 = z_3 = z_4 = 0\} \cup \{z_1 = z_2 = z_3 = z_4 = 0\} \\ &= \{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\}. \end{aligned} \quad (3.2)$$

We now subtract  $Z(\Sigma)$  from  $\mathbb{C}^4$ .

**Step 3:** In this step we have to define an action of  $(\mathbb{C}^*)^m$  on  $\mathbb{C}^4 - Z(\Sigma)$ , where we have to determine  $m$ . For this purpose we find linear combinations of  $\{v_i\}$  with

$$\sum_i q_i v_i = 0.$$

<sup>7</sup>In the literature it is sometimes called the **Stanley-Reisner ideal** or the **variety of the irrelevant ideal**.

<sup>8</sup>The exceptional set encodes information about cones in the fan, i.e. the choice of cones as described in § 3.1.1.

The coefficients  $q_i$  are called **charges**<sup>9</sup>. See § A.1 for more details. In our example, we have two linearly independent charge vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

It is obvious that the charge vectors are not unique.

$$\begin{aligned} & \sum_i q_{ij} v_i = 0 \quad \text{for } j = 1, \dots, \ell \\ \Rightarrow & \sum_{j=1}^{\ell} \alpha_j \left( \sum_i q_{ij} v_i \right) = \sum_i \left( \sum_{j=1}^{\ell} \alpha_j q_{ij} \right) v_i \\ = & \sum_i w_i v_i = 0, \end{aligned}$$

where  $q_{ij}$  denotes the  $i$ -th component of the  $j$ -th charge vector. If we have one set of charges  $\{q_{ij}\}$ , we can use another set of charges whose charge vectors have the form  $w_i = \sum_j \alpha_j q_{ij}$  as long as they are linearly independent. The resulting toric variety will be the same. We conveniently write the generators of the one-dimensional cones and the charges in one matrix denoted by  $(V | Q)$

$$(V | Q) = \left( \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{array} \right).$$

Once the charges are determined, we can define the  $(\mathbb{C}^*)^m$ -action. Since we have two charge vectors,  $m = 2$ . The  $i$ -th charge vector defines the  $i$ -th  $\mathbb{C}^*$ -action:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \sim \begin{pmatrix} \lambda^1 z_1 \\ \lambda^1 z_2 \\ \lambda^0 z_3 \\ \lambda^0 z_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \sim \begin{pmatrix} \mu^0 z_1 \\ \mu^0 z_2 \\ \mu^1 z_3 \\ \mu^1 z_4 \end{pmatrix} \quad (3.3)$$

**Step 4:** We finally obtain the toric variety  $X_{\Sigma}$

$$X_{\Sigma} = \frac{\mathbb{C}^4 - Z(\Sigma)}{(\mathbb{C}^*)^2}.$$

From (3.3) and (3.2) it is clear that

$$X_{\Sigma} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

In § A.1 we discuss this construction in more detail.

We look at other easy examples. Figure 3.6(c) shows a fan of dimension 1. We start with  $\mathbb{C}^2$  since we have two vectors. The exceptional set is clearly  $\{z_1 = z_2 = 0\}$ . The only charge vector is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

---

<sup>9</sup>If the vectors  $\{v_i\}$  are linearly independent, there will not be any charges. We will treat these cases in § 6.2.

Thus the resulting toric variety is  $\mathbb{P}^1$ . Analogously, Figure 3.6(d) shows the fan of  $\mathbb{P}^2$ . In general, the  $(V | Q)$  matrix for  $\mathbb{P}^n$  is

$$(V | Q) = \left( \begin{array}{ccc|c} 1 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \\ -1 & \cdots & -1 & 1 \end{array} \right) = \left( \begin{array}{ccc|c} & & & 1 \\ & \mathbf{1}_n & & 1 \\ -1 & \cdots & -1 & 1 \end{array} \right),$$

where  $\mathbf{1}_n$  denotes the  $n \times n$  identity matrix. The cones generated by every possible combination of  $n$  vectors are included in the fan. Thus we subtract only the origin from  $\mathbb{C}^{n+1}$  because  $v_i$  altogether, i.e.  $(n+1)$  vectors, do not span a cone. We see also that every homogeneous coordinate is scaled by the same factor as it should be.

We look at the last example. Let Figure 3.6(b) be the fan  $\Sigma$ , i.e.

$$(V | Q) = \left( \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & -n & 1 & 0 \\ 0 & 1 & n & 1 \\ 0 & -1 & 0 & 1 \end{array} \right), \quad (3.4)$$

where  $n \in \mathbb{N}$ . Note that if  $n = 0$  we get  $\mathbb{P}^1 \times \mathbb{P}^1$ . Analogously to  $\mathbb{P}^1 \times \mathbb{P}^1$ , the exceptional set is

$$Z(\Sigma) = \{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\}.$$

The  $(\mathbb{C}^*)^2$ -action on  $\mathbb{C}^4 - Z(\Sigma)$  is

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \sim \begin{pmatrix} \lambda^1 z_1 \\ \lambda^1 z_2 \\ \lambda^n z_3 \\ \lambda^0 z_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \sim \begin{pmatrix} \mu^0 z_1 \\ \mu^0 z_2 \\ \mu^1 z_3 \\ \mu^1 z_4 \end{pmatrix}.$$

These complex surfaces parametrized by  $n$  are called **Hirzebruch surfaces**  $\mathbb{F}_n$ . There is another description of  $\mathbb{F}_n$  as follows [BPvdV84, p.140] or [Ful93, p.8 and p.131]

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

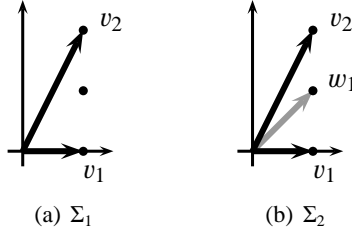
where  $\mathbb{P}(\cdot)$  denotes the projectivization over  $\mathbb{C}^*$  of the argument, e.g.  $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ . We denote the trivial line bundle over  $\mathbb{P}^1$  by  $\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(0)$  and the  $n$ -th power of the hyperplane bundle over  $\mathbb{P}^1$  by  $\mathcal{O}_{\mathbb{P}^1}(n)$  as in § 2.2.

### 3.1.3 Properties of Toric Varieties

We describe in this section how one can read off properties like compactness and smoothness from the fan of the toric variety.

First, we discuss *compactness*. A toric variety is compact if the fan  $\Sigma$  fills the whole  $N_{\mathbb{R}}$ , i.e. if the union of all cones in  $\Sigma$  is equal to  $N_{\mathbb{R}}$ . We write

$$|\Sigma| = N_{\mathbb{R}}. \quad (3.5)$$



**Figure 3.7:** Fans of  $\mathbb{C}^2/\mathbb{Z}^2$  and  $\text{Res}(\mathbb{C}^2/\mathbb{Z}^2)$

The four simple examples discussed in § 3.1.2 are all compact.

Another important property to discuss is the *smoothness*. A toric variety is smooth if every cone in the fan is generated by linearly independent vectors and if the vectors can be completed to a  $\mathbb{Z}$ -basis of  $N$ . This makes the task to resolve a singularity in a toric variety comparably easy.

For a simple example see Figure 3.7. First, consider the toric variety  $X_{\Sigma_1}$  constructed from  $\Sigma_1$ . We have one two-dimensional cone  $\sigma$  and two one-dimensional cones. Note that  $X_{\Sigma_1}$  is not compact since (3.5) is not satisfied. Furthermore the two vectors  $v_1$  and  $v_2$  do not constitute a  $\mathbb{Z}$ -basis for  $\mathbb{Z}^2$ . Thus  $X_{\Sigma_1}$  is singular. It is actually the orbifold  $\mathbb{C}^2/\mathbb{Z}_2$ . We resolve the singularity by adding a generator  $w_1$ , i.e. by **subdividing**  $\sigma$  in two cones  $\tau_1$  and  $\tau_2$ . Now  $\Sigma_2$  has two two-dimensional cones and their generators, i.e.  $\{v_1, w_1\}$  and  $\{v_2, w_1\}$  form both a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^2$ . This is the blowup of the orbifold  $\mathbb{C}^2/\mathbb{Z}_2$ . We will describe non-compact Calabi-Yau orbifolds and their resolutions in § 6.

This gives a rather simple criterion to test whether a fan will yield a smooth toric variety [Gre96, p.111]. We take a maximally dimensional cone and its generators. First, test whether the generators are linearly independent. If they are, then calculate the determinant of the matrix formed by the generators. Do this for every maximally dimensional cone and if the absolute value of all determinants is equal to 1, the resulting toric variety will be smooth.

### 3.1.4 Divisors

In a toric variety the description of divisors is very simple and intuitive.

Let  $\Sigma$  be a fan and  $v_i$  the vectors in  $\Sigma$  for  $i = 1, \dots, r$ . We denote the toric variety constructed from  $\Sigma$  by  $X$ . We assign a homogeneous coordinate  $z_i$  to each  $v_i$ . As explained in § A.6, there is an one-to-one correspondence between  $v_i$  and divisors  $D_i$  of  $X$ . Let us consider  $D_1$ . We have an isomorphism [Cox93, § 1]

$$\Gamma(X, [D_1]) \cong \left\{ z_1 \cdot \sum_{m \in P_{D_1} \cap M} \left( a_m \prod_i z_i^{\langle m, v_i \rangle} \right) \right\},$$

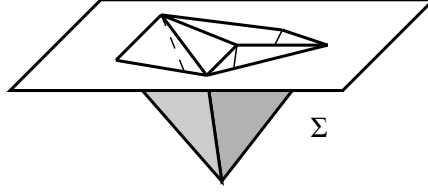
where  $\Gamma(X, [D_1])$  denotes the set of global sections on  $[D_1]$  and

$$P_{D_1} = \{m \in M_{\mathbb{R}} \mid \langle m, v_1 \rangle \geq -1 \text{ and } \langle m, v_i \rangle > 0 \text{ for } i \geq 2\}.$$

As shown in (A.7), the  $\sum_m (a_m \prod_i z_i^{\langle m, v_i \rangle})$  part yields a divisor linearly equivalent to zero. Thus

$$D_1 = \{z_1 = 0\}.$$

This means that for each  $v_i$  there is a divisor corresponding  $D_i$  which is defined by the vanishing of the associated coordinate  $z_i$ . The linear equivalences between  $D_i$  are given by the components of  $v_i$  as



**Figure 3.8:** Fan of a Calabi-Yau toric variety

follows

$$\sum_{i=1}^r v_{ij} D_i \sim 0 \quad \text{for } j = 1, \dots, n,$$

where  $v_{ij}$  denotes the  $j$ -th component of the  $i$ -th vector and  $n$  the dimension of the lattice. In § A.2 it is explained how this relation comes about. Let  $Q_i$  denote the  $i$ -th row of  $(Q)$ . As also explained in § A.2, the rows  $Q_i$  can be seen as formal coordinates of  $D_i$ .

## 3.2 Calabi-Yau Manifolds in Toric Geometry

Toric geometry would not play such a big role in string theory if it could not produce Calabi-Yau manifolds. We will see two different ways how one can obtain Calabi-Yau manifolds in toric geometry. These two methods produce non-compact and compact varieties and both are very useful in string theory.

### 3.2.1 Toric Varieties as Calabi-Yau

Since topological and geometrical properties depend on the fan of a toric variety, we should be able to read off the Calabi-Yau condition from the fan. Indeed, given the one-dimensional generators of the fan, we can easily see whether the resulting variety will be Calabi-Yau or not.

A toric variety  $X$  is Calabi-Yau if all one-dimensional generators lie on a hyperplane. Without loss of generality, the matrix  $(V)$  has the form

$$\begin{pmatrix} v_{11} & \cdots & v_{1,m-1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n1} & \cdots & v_{n,m-1} & 1 \end{pmatrix}. \quad (3.6)$$

The canonical divisor of  $X$  is given by [Ful93, p.85]

$$K_X = - \sum_{i=1}^n D_i,$$

where  $D_i$  is the divisor corresponding to the vector  $v_i$ . Using  $K_X$  and the linear equivalences described in § 3.1.4, we can easily see why toric varieties with  $(V)$  of the form (3.6) are Calabi-Yau:

$$\sum_{i=1}^n v_{i,m} D_i \sim 0 \Rightarrow \sum_{i=1}^n D_i = -K_X \sim 0.$$

Thus  $KX$  is trivial. Figure 3.7 and 3.8 show such fans. As one can see, this kind of toric varieties are non-compact. Thus it seems that it is of limited use for string theory and especially for model building. However, for resolving singularities of orbifold models, this can be indeed very useful. Local models of orbifolds can be described as toric varieties. In § 6 we will describe how to construct non-compact Calabi-Yau orbifolds, how to resolve them and how one can obtain the intersection numbers of divisors.

### 3.2.2 Calabi-Yau Hypersurfaces in Toric Varieties

Since *compact* Calabi-Yau  $n$ -folds are needed for F-theory compactification, we have to do something else. There is a method introduced by V. Batyrev [Bat94]. To obtain a compact Calabi-Yau manifold, we take a compact toric variety and construct a hypersurface which is Calabi-Yau.

We first define the main object for this construction, the **reflexive polyhedron**  $\Delta \in M_{\mathbb{R}}$  and its **dual polyhedron**<sup>10</sup>  $\Delta^* \in N_{\mathbb{R}}$ . A polyhedron is called **reflexive** if it contains only the origin as the integral **interior** point. A point  $p \in \Delta$  is called interior if  $p$  is not lying on any face of  $\Delta$ . The dual polyhedron is defined as follows

$$\Delta^* = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \ \forall y \in \Delta\}. \quad (3.7)$$

One can show that  $\Delta$  is reflexive iff  $\Delta^*$  is reflexive and that  $\Delta = (\Delta^*)^*$  [Bat94, Thm. 4.1.6]. The pair  $(\Delta, \Delta^*)$  is called a **reflexive pair**<sup>11</sup>. The toric varieties constructed from reflexive polyhedrons are called **toric Fano varieties** [Voi99, Def. 4.11 and Prop. 4.14].

Let  $\Delta \in M_{\mathbb{R}}$  be a reflexive polyhedron and  $\Delta^* \in N_{\mathbb{R}}$  its dual<sup>12</sup>. We construct the toric variety from the associated fan  $\Sigma(\Delta^*)$ . Let  $r$  be the number of one-dimensional cones in  $\Sigma(\Delta^*)$ , i.e. we have  $r$  homogeneous coordinates  $z_i$ . Let  $v_i$  denote the generators of one-dimensional cones. The hypersurface defined by the following polynomial

$$p = \sum_{m \in \Delta \cap M} \left( a_m \prod_{i=1}^r z_i^{\langle m, v_i \rangle + 1} \right) \quad (3.8)$$

is then Calabi-Yau. Now we look at the polynomial  $p$  in detail. The sum runs over all integral points  $m$  in  $\Delta$ . The product runs over all  $v_i$ . Note that the power of each  $z_i$  in the product is non-negative because of (3.7). See appendix A.3 for arguments leading to the form of  $p$ . It is shown there that  $p$  is a section of the anticanonical bundle of  $X$ . The coefficient in front of the monomials  $\prod_k z_k^{\langle \cdot, v_k \rangle + 1}$  can be chosen at will like the coefficients of the quintic in  $\mathbb{P}^4$ . As we will see in § 5.4, the coefficients  $a_i$  form a part of the complex structure moduli of the resulting hypersurface.

There is one important issue to be discussed. Many toric Fano varieties and so their Calabi-Yau hypersurfaces obtained this way are singular. Thus we want to at least partly desingularize the hypersurfaces<sup>13</sup>. Let  $X$  be a singular toric Fano variety constructed from  $\Sigma$  and  $X'$  a desingularization of  $X$ . As we will momentarily see, the desingularization procedure will add exceptional divisors to  $X$ , i.e. additional vectors to  $\Sigma$ . Let  $\Sigma'$  denote the fan of  $X'$ . We have a natural projection

$$\tau : \Sigma' \longrightarrow \Sigma$$

of fans. The map  $\tau$  induces a holomorphic map [Oda85, Thm. 1.13]

$$\tau_* : X' \longrightarrow X.$$

We call a desingularization **crepant** if

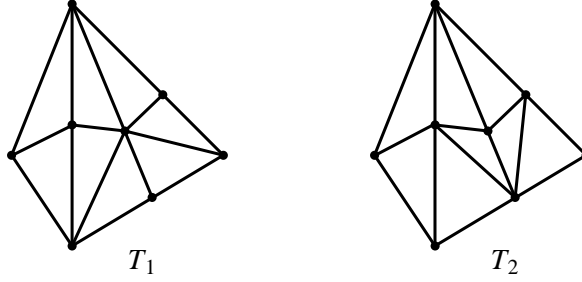
$$(\tau_*)^*(KX) = KX', \quad (3.9)$$

<sup>10</sup>The dual polyhedron is sometimes called the **polar polyhedron**.

<sup>11</sup>In the literature the pair  $(\Delta, M)$  is sometimes called reflexive pair and of course  $(\Delta^*, N)$ , too.

<sup>12</sup>It would be more natural to denote the polyhedron in  $N_{\mathbb{R}}$  by  $\Delta$ . However the other way around is the most common convention which we also follow.

<sup>13</sup>We will remark later why we cannot completely desingularize it.



**Figure 3.9:** Triangulation of a polyhedron

i.e. if the pulled-back canonical bundle under  $\tau_*$  of the singular variety is equal to the canonical bundle of the desingularization. One can show that for toric Fano varieties there exists a *maximal* crepant desingularization which is called the **maximal projective crepant partial** desingularization, **MPCP**-desingularization in the literature [Bat94, Thm. 2.2.24]. For this kind of desingularization we take all integral points of  $\Delta^*$ , i.e.  $\Delta^* \cap N$ , and construct the fan. However, ambiguities will occur in general. Figure 3.9 illustrates an example. Let's assume, the points drawn in Figure 3.9 are integral points of the polyhedron. Then there are different possibilities to define cones over the faces. Two of those so-called **triangulations**  $T_1$  and  $T_2$  are depicted in Figure 3.9. The two triangulations will obviously yield different fans. Thus after we have identified all integral points of  $\Delta^*$ , we have to choose a triangulation<sup>14</sup> to obtain  $\Sigma(\Delta^*)$ . In general, the resulting toric variety  $X'$  will still contain singularities, even after MPCP-desingularization. However, toric Fano varieties will contain only quotient singularities, i.e. orbifold singularities [Bat94, Prop. 2.2.2, Def. 2.2.13, and Thm. 2.2.24]. We again take the polynomial  $p'$  of the form (3.8). This time, we have more generators because the desingularization added a number of generators of one-dimensional cones. The resulting hypersurface  $Z'$  will be in general smooth for ambient toric varieties of dimension 4 [Bat94, Cor. 4.2.3], i.e. Calabi-Yau threefolds constructed this way will be in general smooth. For higher dimensions, e.g. for Calabi-Yau fourfolds, the resulting hypersurfaces will contain orbifold singularities in general since the ambient toric variety contains orbifold singularities.

Note that Batyrev's construction is manifestly mirror symmetric. One of the main consequences of mirror symmetry is the symmetry between  $h^{n-1,1}$  and  $h^{1,1}$  of Calabi-Yau  $n$ -folds. Let  $Z$  be a Calabi-Yau  $n$ -fold constructed from a MPCP-desingularized fan associated to a reflexive polyhedron  $\Delta^* \in N_{\mathbb{R}}$ . Batyrev has given combinatorial formulae for the Hodge numbers [Bat94, Thm. 4.3.7 and Thm. 4.4.2]:

$$\begin{aligned}
 h^{n-1,1}(Z) &= l(\Delta) - n - 1 - \sum_{\text{codim}\Theta=1} l^*(\Theta) + \sum_{\text{codim}\Theta=2} l^*(\Theta) \cdot l^*(\Theta^*), & (3.10) \\
 h^{1,1}(Z) &= l(\Delta^*) - n - 1 - \sum_{\text{codim}\Theta^*=1} l^*(\Theta^*) + \sum_{\text{codim}\Theta=2} l^*(\Theta^*) \cdot l^*(\Theta).
 \end{aligned}$$

We now explain each term in the formulae:  $\Theta$  and  $\Theta^*$  denote a face of  $\Delta$  and a face of  $\Delta^*$  respectively. We denote the number of integral points in  $\Theta$  by  $l(\Theta)$  and the number of integral points in the interior of  $\Theta$  by  $l^*(\Theta)$ . Since  $\Delta$  is also reflexive, let  $\tilde{Z}$  denote the Calabi-Yau  $n$ -fold constructed from  $\Delta$ . As one can easily see

$$h^{n-1,1}(Z) = h^{1,1}(\tilde{Z}) \quad \text{and} \quad h^{1,1}(Z) = h^{n-1,1}(\tilde{Z}).$$

One could ask why we only look at reflexive polyhedrons. To obtain a Calabi-Yau manifold as a hypersurface, we “just” have to take a section of the anticanonical bundle and the second adjunction formula does the job. What is so special about reflexive polyhedrons? The point is that toric Fano

<sup>14</sup>The triangulation has to satisfy number of conditions to yield a crepant desingularization [Bat94, Def. 2.2.15].

varieties have special type of singularities such that after MPCP-desingularization described above, we end up with “mild” orbifold singularities. If the singularities were too worse, we could not even apply the adjunction formula. Furthermore, among other interesting properties the anticanonical divisor of toric Fano varieties is Cartier [Voi99, Def. 4.11], thus we can talk about anticanonical bundle and sections of it. It is now at least plausible why we only look at toric Fano varieties and the crepant desingularization: we want to preserve these “good” properties of the anticanonical divisor and do not want to destroy them by desingularization.

# Sheaves in Algebraic Geometry

*Geometry is magic that works.*  
René Thom, [Tho72]

This chapter serves as a preparation for the discussion of brane moduli and complex structure moduli. Since sheaves and sheaf cohomology play an essential role in algebraic geometry, they are unavoidable for the discussion of the moduli. We introduce only the minimal amount of new terms needed for the discussion of the moduli.

## 4.1 Sheaves

One cannot think of the modern algebraic geometry without sheaves. Here we give the definition of a sheaf and look at several examples. For large part we follow [GH78, § 0.3].

A sheaf<sup>1</sup> is an abstract way of putting “things”, e.g.  $\mathcal{C}^\infty$ -functions, holomorphic functions etc., on a topological space. Although sheaves can be defined over general topological spaces, we restrict ourselves to sheaves over compact manifolds. Let  $M$  be a compact manifold. A **sheaf**  $\mathcal{S}$  assigns to every open set  $U$  of  $M$  an Abelian group denoted by  $\mathcal{S}(U)$  and to any two open sets  $V \subseteq U$  a group homomorphism

$$r_{UV} : \mathcal{S}(U) \longrightarrow \mathcal{S}(V)$$

which is called the **restriction map**. The restriction map is conveniently written  $\sigma|_V$  for  $\sigma \in \mathcal{S}(U)$  and  $V \subseteq U$ . The restriction map has to satisfy three conditions:

1. For any three open sets  $W \subseteq V \subseteq U$ , we have  $r_{UW} = r_{VW} \circ r_{UV}$ .
2. For  $\sigma \in \mathcal{S}(U \cup V)$  with  $\sigma|_U = \sigma|_V = 0$ , i.e.  $r_{U \cup V, U}(\sigma) = r_{U \cup V, V}(\sigma) = 0$ , we have  $\sigma = 0$ .
3. For any two open sets  $U_1$  and  $U_2$  and two elements of the corresponding Abelian groups  $\sigma_1 \in \mathcal{S}(U_1)$  and  $\sigma_2 \in \mathcal{S}(U_2)$  with  $\sigma_1|_{U_1 \cap U_2} = \sigma_2|_{U_1 \cap U_2}$ , i.e.  $r_{U_1, U_1 \cap U_2}(\sigma_1) = r_{U_2, U_1 \cap U_2}(\sigma_2)$ , we have unique<sup>2</sup> element  $\sigma \in \mathcal{S}(U_1 \cup U_2)$  such that  $\sigma|_{U_i} = \sigma_i$ .

The Abelian group  $\mathcal{S}(U)$  is also called the **set of sections** of  $\mathcal{S}$  over  $U$ .

Now we look at some examples. The first example is  $\mathcal{C}^\infty$ , the sheaf of  $\mathcal{C}^\infty$ -functions. The set of holomorphic and nonzero holomorphic functions  $\mathcal{O}$  and  $\mathcal{O}^*$  are also examples of sheaves and likewise  $\mathcal{M}$  and  $\mathcal{M}^*$ , sheaves of meromorphic and not identically zero meromorphic functions. The sheaf of holomorphic functions of  $M$  is also called the **structure sheaf** and is often denoted by  $\mathcal{O}_M$ . For our purpose, the discussion of the brane moduli, the sheaf of local  $\mathcal{C}^\infty$ -forms of degree  $(p, q)$  denoted by  $\mathcal{A}^{p,q}$  is the most important example. For more examples see [GH78, p.36].

<sup>1</sup>It's a translation of the French word **faisceau**. In German it is translated as **Garbe**.

<sup>2</sup>The uniqueness is a consequence of the second condition.

## 4.2 Cohomology of sheaves

There are many ways<sup>3</sup> to obtain cohomology of sheaves. We look at two of them: Čech cohomology and cohomology through fine resolutions.

### 4.2.1 Čech cohomology

Sheaf cohomology can be defined purely combinatorially, see for example [GH78, p.38], [BT82, p.110], or [Kod86, § 3.3.(b)]. Let  $M$  be a compact manifold and  $\mathcal{S}$  a sheaf on  $M$ . Let  $\mathfrak{U} = \{U_i\}_{i \in I}$  be an open cover<sup>4</sup> of  $M$ . We call a set

$$\{\sigma_i \in \mathcal{S}(U_i) \mid \forall i \in I\}$$

a **0-cochain**  $c^0$  respect to  $\mathfrak{U}$ . Thus  $c^0$  assigns to each  $U_i$  an element  $\sigma_i \in \mathcal{S}(U_i)$ . A **1-cochain**  $c^1$  is a set

$$\{\sigma_{ij} \in \mathcal{S}(U_i \cap U_j) \mid \forall i, j \in I \text{ with } U_i \cap U_j \neq \emptyset\}$$

with  $\sigma_{ij} = -\sigma_{ji}$ . Let  $U_{i_0 \dots i_p}$  denote the intersection  $U_{i_0} \cap \dots \cap U_{i_p}$ . Generally, a  **$p$ -cochain**  $c^p$  is a set

$$\{\sigma_{i_0 \dots i_p} \in \mathcal{S}(U_{i_0 \dots i_p}) \mid \forall i_0, \dots, i_p \in I \text{ with } U_{i_0 \dots i_p} \neq \emptyset\},$$

where  $\sigma_{i_0 \dots i_p}$  are antisymmetric in all indices  $i_0, \dots, i_p$ . The set of all  $p$ -cochains with respect to  $\mathfrak{U}$  we denote by  $C^p(\mathfrak{U}, \mathcal{S})$ .

We now want to define a map analogous to  $d$  for differential forms to be able to do cohomology theory. Therefore, we define the **coboundary** map  $\delta^{p-1}$  for  $p \geq 0$

$$\delta^{p-1} : C^{p-1}(\mathfrak{U}, \mathcal{S}) \longrightarrow C^p(\mathfrak{U}, \mathcal{S}).$$

Let  $c^{p-1} = \{\sigma_{i_1 \dots i_p}\}$  be a  $(p-1)$ -cochain, then

$$\begin{aligned} \delta^{p-1} c^{p-1} &= \delta^{p-1} \{\sigma_{i_1 \dots i_p}\} = \sum_{j=0}^p (-1)^j \sigma_{i_0 \dots \hat{i}_j \dots i_p} \Big|_{U_{i_0 \dots i_p}} \\ &= \{\tau_{i_0 \dots i_p}\} \\ &= c^p \in C^p(\mathfrak{U}, \mathcal{S}), \end{aligned}$$

where the index with a hat is omitted. Note that the index of  $\sigma$  goes from 1 to  $p$ , i.e. we have  $p$  indices. For  $\tau$  the index goes from 0 to  $p$ , thus we have  $p+1$  indices. Because of the factor  $(-1)^j$ , the resulting  $p$ -cochain is antisymmetric in its indices. It follows that  $(\delta^{p+1} \circ \delta^p)c^p = 0$  for all  $c^p$ . Thus we obtain a **cochain complex**

$$C^0(\mathfrak{U}, \mathcal{S}) \xrightarrow{\delta^0} C^1(\mathfrak{U}, \mathcal{S}) \xrightarrow{\delta^1} C^2(\mathfrak{U}, \mathcal{S}) \xrightarrow{\delta^2} \dots$$

As usual, we consider the set of exact  $p$ -cochains  $Z^p(\mathfrak{U}, \mathcal{S})$  which is the kernel of  $\delta^p$ , i.e.,

$$Z^p(\mathfrak{U}, \mathcal{S}) = \{c^p \in C^p(\mathfrak{U}, \mathcal{S}) \mid \delta^p c^p = 0\} = \text{Ker } \delta^p.$$

The  $q$ -th cohomology group  $H^q(\mathfrak{U}, \mathcal{S})$  is defined as follows

$$H^q(\mathfrak{U}, \mathcal{S}) = Z^q(\mathfrak{U}, \mathcal{S}) / \delta^{q-1} C^{q-1}(\mathfrak{U}, \mathcal{S}) = \text{Ker } \delta^q / \text{Im } \delta^{q-1}.$$

<sup>3</sup>See [Har77, p.201] for a list and references for each approach.

<sup>4</sup>To be more precise,  $\mathfrak{U}$  has to be **locally finite**. See for example [Kod86, § 2.1.(c)]. However, since we consider only compact manifolds, this is irrelevant.

Note that we set  $H^0(\mathfrak{U}, \mathcal{S}) = Z^0(\mathfrak{U}, \mathcal{S})$ . The zeroth cohomology group can be described in a different way. Let  $c^0 = \{\sigma_i\} \in Z^0(\mathfrak{U}, \mathcal{S})$ . Then

$$\sigma_i - \sigma_j = 0 \Rightarrow \sigma_i = \sigma_j \quad \text{in } U_{ij} \quad \text{for } U_{ij} \neq \emptyset.$$

This simply means that  $c^0$  is a global section of  $\mathcal{S}$  on  $M$ . Thus

$$H^0(\mathfrak{U}, \mathcal{S}) = \Gamma(M, \mathcal{S}).$$

Everything so far has been defined for a fixed<sup>5</sup>  $\mathfrak{U}$ . However, one can obtain well defined cohomology groups of  $M$  with coefficients in  $\mathcal{S}$  denoted by  $H^q(M, \mathcal{S})$  which do not depend on  $\mathfrak{U}$ . One considers a limiting process, where  $\mathfrak{U}$  becomes “infinitely fine” [Kod86, p.117]. We call  $H^i(M, \mathcal{S})$  the  $i$ -th sheaf cohomology group or **Čech cohomology group**<sup>6</sup>.

#### 4.2.2 Fine Resolutions

Let  $M$  be a compact complex manifold and  $E$  a complex vector bundle over  $M$ . Let  $\mathcal{O}_M(E)$  denote the sheaf of local holomorphic sections on  $E$ . It will be crucial to be able to determine the cohomology groups  $H^q(M, \mathcal{O}_M(E))$ . Since there is no worry for confusion, we write  $H^i(M, E)$  for  $H^i(M, \mathcal{O}_M(E))$ . Fine resolution is a convenient method to calculate these groups. We first discuss fine sheaves, fine resolutions, and then the Dolbeault theorem [Hir78, § 2.11 and § 15] or [Kod86, § 3.4].

We will not give a precise definition of a **fine sheaf**, but refer to [Hir78, § 2.11] or [Kod86, Def. 3.13]. It is sufficient for us that  $\mathcal{S}^{p,q}(M)$  are fine sheaves. There is one important property of fine sheaves which is relevant. Let  $\mathcal{S}$  be a fine sheaf. Then

$$H^n(M, \mathcal{S}) = 0 \quad \forall n \geq 1. \quad (4.1)$$

Note that for a sheaf to be fine implies this property, but in general not vice versa.

An exact sequence of sheaves

$$0 \longrightarrow \mathcal{S} \xrightarrow{h} \mathcal{S}_0 \xrightarrow{h^0} \mathcal{S}_1 \xrightarrow{h^1} \mathcal{S}_2 \xrightarrow{h^2} \dots \quad (4.2)$$

is called a **resolution** of  $\mathcal{S}$  if

$$H^q(M, \mathcal{S}_p) = 0 \quad \text{for } q \geq 1 \quad \text{and } p \geq 0.$$

A resolution is called **fine** if  $\mathcal{S}_p$  are fine sheaves for  $p \geq 0$ . It is therefore clear that if  $\mathcal{S}_p$  are fine for all  $p \geq 0$  and we have an exact sequence of form (4.2), then this sequence is a resolution because of (4.1). We apply  $\Gamma(M, \cdot)$  to the fine resolution (4.2) to obtain

$$0 \longrightarrow \Gamma(M, \mathcal{S}) \xrightarrow{h_*} \Gamma(M, \mathcal{S}_0) \xrightarrow{h_*^0} \Gamma(M, \mathcal{S}_1) \xrightarrow{h_*^1} \Gamma(M, \mathcal{S}_2) \xrightarrow{h_*^2} \dots, \quad (4.3)$$

a sequence of global sections which is in general not exact. This sequence is a cochain complex since  $h_*^{p+1} \circ h_*^p = 0$ . Thus we can consider its cohomology groups. The main point of considering fine resolutions is that surprisingly

$$\begin{aligned} H^0(M, \mathcal{S}) &\cong \text{Ker } h_*^0, \\ H^q(M, \mathcal{S}) &\cong \text{Ker } h_*^q / \text{Im } h_*^{q-1} \quad \text{for } q \geq 1, \end{aligned}$$

<sup>5</sup>However,  $H^0(\mathfrak{U}, \mathcal{S})$  is independent of  $\mathfrak{U}$  since  $H^0(\mathfrak{U}, \mathcal{S}) = \Gamma(M, \mathcal{S})$ .

<sup>6</sup>To be precise, the cohomology of sheaves is defined by injective resolutions, see for example [Har77, Asp04, Ban07]. One can show that these groups are canonically isomorphic to the Čech cohomology groups for paracompact spaces [Ban07, § 1.2.2]. We only consider sheaves on compact spaces. Thus it does not matter for us.

i.e. the  $q$ -th cohomology group of the complex (4.3) obtained from a fine resolution (4.2) is isomorphic to the cohomology group  $H^q(M, \mathcal{S})$  defined as in § 4.2.1. Note that the zeroth cohomology group is not defined as  $\text{Ker } h_*^0 / \text{Im } h_*$  as one might assume, but as  $\text{Ker } h_*^0$ . Thus the cohomology groups of  $\mathcal{S}$  are determined from the complex

$$0 \longrightarrow \Gamma(M, \mathcal{S}_0) \xrightarrow{h_*^0} \Gamma(M, \mathcal{S}_1) \xrightarrow{h_*^1} \Gamma(M, \mathcal{S}_2) \xrightarrow{h_*^2} \Gamma(M, \mathcal{S}_3) \xrightarrow{h_*^3} \dots$$

If we have a fine resolution for  $\mathcal{O}_M(E)$ , we can determine its cohomology groups  $H^q(M, E)$  as described above. Do we have a fine resolution for  $\mathcal{O}_M(E)$ ? The answer is yes.

First we consider the case  $E = \wedge^p T^*M$ . As mentioned above,  $\mathcal{A}^{p,q}(M)$  are fine. Furthermore the so-called  **$\bar{\partial}$ -Poincaré lemma** [GH78, p.25] tells us that

$$0 \longrightarrow \mathcal{O}_M(\wedge^p T^*M) \xrightarrow{\iota} \mathcal{A}^{p,0}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{(M)^{p,1}} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2}(M) \xrightarrow{\bar{\partial}} \dots \quad (4.4)$$

is exact, where the map  $\iota$  is the embedding of  $\mathcal{O}_M(\wedge^p T^*M)$  in  $\mathcal{A}^{p,0}(M)$ . We call (4.4) the **Dolbeault sequence**. Thus we have a fine resolution for  $\mathcal{O}_M(\wedge^p T^*M)$  and the cohomology groups  $H^q(M, \wedge^p T^*M)$  can be obtained from the complex

$$0 \longrightarrow A^{p,0}(M) \xrightarrow{\bar{\partial}_*} A^{p,1}(M) \xrightarrow{\bar{\partial}_*} A^{p,2}(M) \xrightarrow{\bar{\partial}_*} A^{p,3}(M) \xrightarrow{\bar{\partial}_*} \dots,$$

where we write  $A^{p,q}(M)$  for  $\Gamma(M, \mathcal{A}^{p,q}(M))$ . This is the **Dolbeault theorem**. Let

$$Z^{p,q}(M) = \{a \in A^{p,q}(M) \mid \bar{\partial}a = 0\}.$$

Then

$$H^q(M, \wedge^p T^*M) = Z^{p,q}(M) / \bar{\partial}A^{p,q-1}(M).$$

The Dolbeault theorem shows also that

$$H^q(M, \wedge^p T^*M) \cong H^{p,q}(M).$$

We can do even more. Let  $E$  be a complex vector bundle on  $M$ . The following sequence is a fine resolution [Hir78, p.119]

$$0 \longrightarrow \mathcal{O}_M(E \otimes \wedge^p T^*M) \longrightarrow \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}} \mathcal{A}(E)^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2}(E) \xrightarrow{\bar{\partial}} \dots, \quad (4.5)$$

where  $\mathcal{A}^{p,q}(E)$  denotes the sheaf of local sections of type  $(p, q)$  with coefficients in  $E$ . For complex structure moduli of  $M$ , the bundle  $E$  will be  $TM$ . We get from (4.5), applying  $\Gamma(M, \cdot)$ , the complex

$$0 \longrightarrow A^{p,0}(E) \xrightarrow{\bar{\partial}_*} A^{p,1}(E) \xrightarrow{\bar{\partial}_*} A^{p,2}(E) \xrightarrow{\bar{\partial}_*} A^{p,3}(E) \xrightarrow{\bar{\partial}_*} \dots,$$

where we write as before  $A^{p,q}(E)$  for  $\Gamma(M, \mathcal{A}^{p,q}(E))$ . Let as usual

$$Z^{p,q}(E) = \{a \in A^{p,q}(E) \mid \bar{\partial}a = 0\}.$$

We obtain the cohomology groups

$$H^q(M, E \otimes \wedge^p T^*M) = Z^{p,q}(E) / \bar{\partial}A^{p,q-1}(E).$$

Setting  $p = 0$ , we finally get

$$H^q(M, E) = Z^{0,q}(E) / \bar{\partial}A^{0,q-1}(E).$$

# F-Theory

*I am your father.*

Darth Vader, Star Wars Episode V: The Empire Strikes Back

We want to do model building within the framework of F-theory [Vaf96]. In this chapter we review the concepts of this particular theory and its orientifold limit. We also discuss the brane moduli of the theory.

## 5.1 Basics of F-Theory

F-theory non-perturbatively describes consistent configurations of type IIB string theory with D7-branes. To motivate F-theory, we first look at the  $\mathrm{SL}(2, \mathbb{Z})$ -symmetry of massless bosonic fields of type IIB string theory. In the NS-NS sector we have the metric  $g_{MN}$ , the  $B_2$ -field and the dilaton  $\phi$ . The vacuum expectation value of the dilaton serves as the string coupling constant  $g_s$  when it is exponentiated. From the R-R sector we have the  $p$ -form fields  $C_p$  with  $p = 0, 2, 4$ . We focus on the fields  $\phi$  and  $C_0$ . We combine the dilaton and  $C_0$  into a complex field  $\tau$

$$\tau = C_0 + ie^{-\phi}$$

which is called the **axiodilaton**. The  $\mathrm{SL}(2, \mathbb{Z})$ -symmetry is conjectured to be an *exact* symmetry of the type IIB string theory. Let  $T$  denote the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

which generates  $\mathrm{SL}(2, \mathbb{Z})$  together with another matrix  $S$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The symmetry group acts as follows

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad F_3 - \tau H_3 \rightarrow \frac{F_3 - \tau H_3}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

where  $F_3$  is the field strength of  $C_2$  and  $H_3$  the field strength of  $B_2$ . The form field  $C_4$  does not transform under the  $\mathrm{SL}(2, \mathbb{Z})$ -symmetry. Let us compactify the ninth and the tenth dimension on  $\mathbb{P}^1$  with the complex coordinate  $z$  since  $\mathbb{P}^1$  is the simplest compact complex manifold. If we assume that the D7-branes fill the non-compact dimensions, it will be a point in the internal manifold. Let  $z_0$  be the coordinate of a D7-brane. D7-branes are magnetically charged under  $C_0$ . This means that

$$C_0 \rightarrow C_0 + 1 \Rightarrow \tau \rightarrow \tau + 1 \tag{5.1}$$

if we go around a path encircling the D7-brane. Thus in the neighborhood of the D7-brane  $\tau$  has the form

$$\tau(z) = \frac{1}{2\pi i} \ln(z - z_0). \quad (5.2)$$

We have a highly non-perturbative situation: The string coupling is not constant, but it even has a logarithmic singularity at the position of the D7-brane. The transformation (5.1) of  $\tau$  when we encircle  $z_0$  is called a **monodromy**. The monodromy can be realized by an  $\text{SL}(2, \mathbb{Z})$  transformation if we act with  $T$  on  $\tau$ . The  $\text{SL}(2, \mathbb{Z})$ -symmetry contains also the strong-weak duality since the generator  $S$  transforms  $\tau$  as follows

$$\tau \rightarrow -\frac{1}{\tau}.$$

A general  $\text{SL}(2, \mathbb{Z})$  element transforms a D7-brane, which has the charge  $(0, 1)$ , into a 7-brane with a general charge  $(p, q)$  with  $p, q \in \mathbb{Z}$ . The  $(p, q)$ -branes are perturbative D-branes on which  $(p, q)$ -strings<sup>1</sup> can end [Vaf96, p.4].

As we can see from (5.2), at the position of the 7-branes the field  $\tau$  shows singular behaviour. Since  $\text{SL}(2, \mathbb{Z})$  is the symmetry group of a torus, we interpret the complex field  $\tau$  as the complex structure modulus of a torus [Vaf96] which means that we have a different torus on every point of the internal manifold. Degenerate tori, i.e. singular fibers, signal the presence of 7-branes. This brings us to the concept of elliptic fibration which will be discussed in detail in the next section. F-theory can be seen as a twelve-dimensional theory which, compactified on an elliptically fibered Calabi-Yau  $n$ -folds for  $n = 1, 2, 3$ , and 4, consistently and non-perturbatively describes the type IIB theory with 7-branes. The additional two dimensions correspond to the torus on each point of the internal manifold. The elliptic fibration is a beautiful way to geometrize the  $\text{SL}(2, \mathbb{Z})$ -symmetry and 7-brane configurations.

## 5.2 Elliptic Fibration

To work with F-theory, it is crucial to understand a special fibration structure, the elliptic fibration. Let  $M$  be a compact complex manifold of complex dimension  $m$ . This manifold will serve as the base manifold of the fibration. In most cases we will have  $m = 3$  to obtain a Calabi-Yau 4-fold.

### 5.2.1 Fibration

First, we abstractly define the **elliptic fibration**. Let  $\pi_E : E \longrightarrow M$  be a holomorphic fibration on  $M$  with an elliptic curve, i.e.  $T^2$ , as its fiber. This means

$$\begin{array}{ccc} T^2 & \longrightarrow & E \\ & & \downarrow \pi_E \\ & & M \end{array}$$

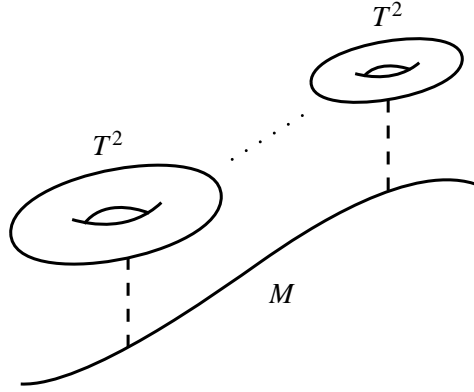
and Figure 5.1 illustrates this.

An elliptic fibration has a section<sup>2</sup>  $\sigma$

$$\begin{array}{ccc} E & & \\ \pi_E \downarrow \curvearrowright \sigma & & \\ M & & \end{array}$$

<sup>1</sup>See [Pol98, § 13.6 and § 14.1] for a treatment of  $(p, q)$ -strings.

<sup>2</sup>Note that a fibration with complex curve of genus one as its fiber *does not* have to have a section. See [DOPW99, § 2.1] or [Sha96, p.206].



**Figure 5.1:** At every point of the base manifold  $M$  we have a torus  $T^2$  as the fiber. The tori will be different from point to point and there will be even degenerate tori.

with  $\pi_E \circ \sigma = \text{id}$ . We naturally identify  $M$  with its image under  $\sigma$ , i.e. the subvariety  $\sigma(M)$  of  $E$ . From now on we write  $M$  instead of  $\sigma(M)$ .

To analyze the elliptic fibration, the above form is nearly useless. Thus we should try to find a more explicit form of it. This is called the **Weierstraß form** of  $E$  which we denote by  $W_E$ . We follow [Kas77]. See for example also [FMW97, § 2.2] or [DHOR04, § 2]. Consider a  $\mathbb{P}^2$ -bundle  $P$  over  $M$

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}(\mathcal{O}_M \oplus L^2 \oplus L^3) = P, \\ & & \downarrow \pi_P \\ & & M \end{array}$$

where we write  $\mathcal{O}_M$  for the trivial line bundle and  $L$  is a line bundle over  $M$ . Here  $\mathbb{P}(\cdot)$  means the projectivization over  $\mathbb{C}^*$  of the argument.

Let furthermore  $\mathcal{O}_P(n)$  denote the  $n$ -th power of the inverse of the tautological bundle<sup>3</sup>  $\mathcal{O}_P(1)$  over  $P$ . Then

$$\mathcal{O}_P(n) \otimes \pi_P^* L^m$$

is a line bundle over  $P$  for any  $n, m \in \mathbb{Z}$ . Note that sections of  $\mathcal{O}_P(n) \otimes \pi_P^* L^m$  can be viewed as polynomials of homogeneous degree  $n$  in  $P$  with coefficients in  $L^m$ .

We can now describe the Weierstraß form of the fibration. It consists of

1. a line bundle  $L$  over  $M$ ,
2. two sections  $f \in \Gamma(M, L^6)$  and  $g \in \Gamma(M, L^6)$ ,

where  $\Gamma(M, L^n)$  denotes the set of global sections from  $M$  to  $L^n$ .

---

<sup>3</sup>Since  $P$  is the projectivization of the three plane bundle  $\mathcal{O}_M \oplus L^2 \oplus L^3$ , the “god-given” tautological bundle is defined analogously to  $\mathbb{P}^n$ .

There are three canonical sections,

$$\begin{aligned} x &\in \Gamma(P, \mathcal{O}_P(1) \otimes \pi_P^* L^2), \\ y &\in \Gamma(P, \mathcal{O}_P(1) \otimes \pi_P^* L^3), \\ z &\in \Gamma(P, \mathcal{O}_P(1)). \end{aligned}$$

These sections restricted to the fiber  $\pi_P^{-1}(m) \cong \mathbb{P}^2$  for  $m \in M$  correspond to the normal three homogeneous coordinates of  $\mathbb{P}^2$ .

The elliptic fibration can be now described as the zero locus of a unique section  $\mu$

$$\mu \in \Gamma(P, \mathcal{O}_P(3) \otimes \pi_P^* L^6),$$

i.e. as a divisor in  $P$ . The section  $\mu$  has the form

$$\mu = y^2 z - x^3 - \pi_P^*(f) x z^2 - \pi_P^*(g) z^3.$$

We can write down the **discriminant** of  $W_E$

$$\Delta = 4f^3 + 27g^2 \in \Gamma(M, L^{12})$$

whose zero locus gives the location, where the fibration becomes singular, i.e. the position of the 7-branes. Since  $\Delta$  only depends on  $f$  and  $g$ , the brane positions, i.e. the brane moduli, depend only on  $f$  and  $g$ . In explicit examples  $f$  and  $g$  will be polynomials and therefore also the section  $\mu$  which we call the **Weierstraß polynomial**. This means that polynomial deformations of the Weierstraß polynomial correspond to the brane moduli.

### 5.2.2 $\mathbb{P}_{1,2,3}$ -Fibration

The  $\mathbb{P}_{1,2,3}$ -bundle description, i.e. the weighted projective space with weights  $(1, 2, 3)$  as fiber, is equivalent to the  $\mathbb{P}_2$ -bundle description [MV96, p.12]. For toric geometry construction of elliptic Calabi-Yau  $n$ -folds the  $\mathbb{P}_{1,2,3}$ -bundle  $P$  is better suited than  $\mathbb{P}^2$ -bundle. We consider the bundle

$$\begin{array}{ccc} \mathbb{P}_{1,2,3} & \longrightarrow & P \\ & & \downarrow \pi_P \\ & & M \end{array}$$

with three sections

$$\begin{aligned} x &\in \Gamma(P, \mathcal{O}_P(2) \otimes \pi_P^* L^2), \\ y &\in \Gamma(P, \mathcal{O}_P(3) \otimes \pi_P^* L^3), \\ z &\in \Gamma(P, \mathcal{O}_P(1)). \end{aligned}$$

The section  $\mu$  whose zero locus describes the elliptic fibration has the form

$$\mu = y^2 - x^3 - \pi_P^*(f) x z^3 - \pi_P^*(g) z^6,$$

where  $f$  and  $g$  are sections in  $L^4$  and  $L^6$ , respectively. The discriminant has the same form as in the  $\mathbb{P}^2$ -bundle case.

### 5.2.3 Elliptically Fibered Calabi-Yau

If we want to obtain an elliptic Calabi-Yau  $E$ , we have to choose the line bundle for the sections  $f$  and  $g$  appropriately. The line bundle  $L^{-1}$  is the normal bundle of  $M$  in  $E$  [Asp96, p.80], i.e.,

$$L^{-1} = N_E M.$$

It is now easy to construct an *elliptically fibered Calabi-Yau*. We apply the second adjunction formula to obtain

$$KM = (KE \otimes [M])|_M$$

from which we get the first Chern class

$$c_1(KM) = c_1(KE) + c_1([M]) = -c_1(L) \Rightarrow c_1(M) = c_1(L) \quad (5.3)$$

since we want  $E$  to be Calabi-Yau. Thus we obtain  $L = KM^{-1}$ . If we choose the line bundle  $L$  to be the anti-canonical bundle of  $M$ , we get an elliptically fibered Calabi-Yau manifold.

### 5.2.4 Elliptic Calabi-Yau Fourfolds in Toric Geometry

For application in the F-theory we need an elliptically fibered Calabi-Yau fourfold. For this construction we follow [KLRY98]. As we have seen, we need a threefold and a  $\mathbb{P}_{1,2,3}$ -bundle over it.

First, we look at how fibration is realized in toric geometry. Consider therefore  $(V)$  of the form

$$\left( \begin{array}{c|c} V_1 & \tilde{V} \\ \hline 0 & V_2 \end{array} \right) \quad (5.4)$$

whose toric variety we call  $X$ . Let further  $X_1$  and  $X_2$  denote the toric varieties constructed from  $V_1$  and  $V_2$  separately. We obtain a fibration of the form

$$\begin{array}{ccc} X_2 & \longrightarrow & X \\ & & \downarrow \\ & & X_1 \end{array}$$

from (5.4), where  $X_1$  is the base and  $X_2$  is the fiber. The components of  $\tilde{V}$  determines the fibration structure. For  $\tilde{V} = 0$  we get a direct product. We have already looked at two examples of this kind,  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathbb{F}_n$ . If we look at the  $(V | Q)$  matrices (3.1) and (3.4), we immediately see such a structure.

Since we want to use  $\mathbb{P}_{1,2,3}$ -fiber,  $(V_2)$  will be of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -3 \end{pmatrix}. \quad (5.5)$$

The charge vector for  $(V_2)$  is

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

which scales the homogeneous coordinates appropriately. If  $(V_1)$  is the matrix of vertices for toric Fano threefolds, one can easily give  $\tilde{V}$  suited for our purpose in terms of  $(V_1)$  and  $(V_2)$  [KLRY98, Sect. 5]. Note that toric Fano threefolds are completely classified up to isomorphism and there are 18 of them

[Oda85, p.90]. Let us denote them by  $F_n$  with  $n = 1, \dots, 18$ . Now we determine  $\tilde{V}$  such that the Calabi-Yau hypersurface in the toric variety constructed from the polyhedron whose vertices are the rows of  $(V)$  is elliptically fibered. Let  $v_{ij}$  with  $j = 1, \dots, 5$  denote the  $j$ -th component of the  $i$ -th row of  $(V_1)$ . Note that we already have fixed these components for  $j = 1, 2, 3$  by choosing the base to be one of the  $F_n$ . We set

$$\begin{aligned} v_{i4} &= -2(v_{i1} + v_{i2} + v_{i3} - 1) \\ v_{i5} &= -3(v_{i1} + v_{i2} + v_{i3} - 1). \end{aligned} \tag{5.6}$$

The resulting Calabi-Yau hypersurface constructed as outlined in § 3.2.2 will be elliptically fibered.

### 5.2.5 Gauge Enhancements and Singularities

In F-theory, we can read off the gauge symmetry enhancements due to 7-branes from the different types of singularities of the elliptic fiber. As it is well known, coincident branes give rise to enhanced gauge symmetries. Colliding singular fibers corresponds to coincident 7-branes and thus one expects enhanced gauge symmetries. For the elliptic K3 Kodaira has given a complete classification of the singular fibers [Kod63, Thm. 6.2] or [MV96, § 3.1]. These singularities are of type  $ADE$ : for example an  $A_n$  singularity gives the corresponding  $A_n$  Lie algebra as the gauge enhancement and analogously other gauge groups corresponding to the Lie algebras  $D_n, E_6, E_7$ , and  $E_8$ . For higher-dimensional compactifications no such classification exists. However, there are efforts to determine the gauge enhancements directly from the polyhedron of Calabi-Yau hypersurfaces in toric varieties [CPR97, PS97, CF98].

### 5.2.6 Elliptic K3

We will now go through every detail of the most simple, yet non-trivial example of elliptic Calabi-Yau, namely elliptic K3. Since the K3 is a Calabi-Yau 2-fold, the base is complex one-dimensional, i.e.  $\mathbb{P}^1$ . Using (5.4), (5.5) and (5.6), we construct the matrix  $(V)$  of the suitable  $\mathbb{P}_{1,2,3}$ -bundle over  $\mathbb{P}^1$  and choose a set of charge vectors  $(Q)$

$$\left( \begin{array}{c|cc} V_1 & \tilde{V} \\ \hline 0 & V_2 \end{array} \right) = \left( \begin{array}{c|cc} 1 & 0 & 0 \\ -1 & -4 & -6 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{array} \right) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} \rightarrow \begin{pmatrix} w_1 \\ w_2 \\ x \\ y \\ z \end{pmatrix} \quad \text{and} \quad (Q) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 4 & 2 \\ 6 & 3 \\ 0 & 1 \end{pmatrix},$$

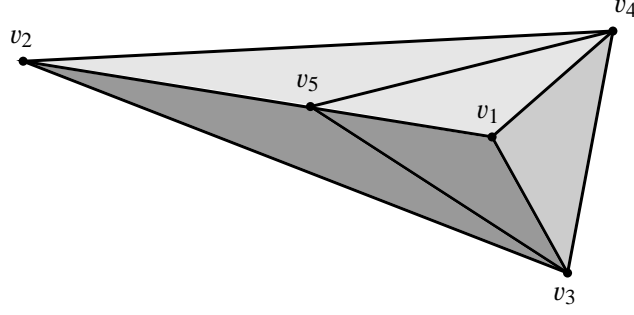
where we have assigned the coordinates  $(w_1, w_2, x, y, z)$  to the vectors. The vertices of the reflexive polyhedron  $\Delta^*$  are as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

All integral points of  $\Delta^*$  are given in (A.10). We now construct the fan  $\Sigma(\Delta^*)$ . For this purpose we have to choose a triangulation because  $v_5$  lies on a face of  $\Delta^*$ . However, as Figure 5.2 shows, there is one unique triangulation. Note that the origin being an interior point is not used for the triangulation. Table 5.1 shows all six three-dimensional cones of  $\Sigma(\Delta^*)$ . All the faces of those three-dimensional cones are also contained in  $\Sigma(\Delta^*)$ . The exceptional set is as follows

$$Z(\Sigma(\Delta^*)) = \{w_1 = w_2 = 0\} \cup \{x = y = z = 0\}.$$

We construct the toric variety  $X_{\Sigma(\Delta^*)}$  from  $\Sigma(\Delta^*)$  as outlined in § 3.1.2. It is clear that  $X_{\Sigma(\Delta^*)}$  contains singularities since there are integral points of the polyhedron not taken into account. One can also check



**Figure 5.2:** Triangulation of  $\Delta^*$  of K3. The vectors  $v_1, v_2, v_3,$  and  $v_4$  form a tetrahedron with  $v_1$  at the top. The vector  $v_5$  lies on the line between  $v_1$  and  $v_2$ .

$(v_1, v_3, v_5)$	$(v_2, v_3, v_5)$
$(v_1, v_3, v_4)$	$(v_2, v_4, v_5)$
$(v_1, v_4, v_5)$	$(v_2, v_3, v_4)$

**Table 5.1:** Three-dimensional cones of  $\Sigma(\Delta^*)$  of K3

this directly by calculating the determinant of the generators of the three-dimensional cones.

To obtain the polynomial defining the Calabi-Yau hypersurface, we need the dual reflexive polyhedron  $\Delta$ . The vertices of  $\Delta$  are given by

$$\begin{pmatrix} 11 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Although the calculation of the dual polyhedron and its integral points can be done by hand in a finite amount of time, even for higher-dimensional polyhedrons, we use the computer program PALP of M. Kreuzer and H. Skarke [KS04]. There are 39 integral points of  $\Delta$  including the vertices. These points are listed in (A.11). All possible monomials are listed in Table 5.2. Thus the polynomial defining the Calabi-Yau hypersurface is

$$\begin{aligned} p &= A_1 y^2 + A_2 x^3 + \left( \sum_{i=0}^4 c_i w_1^i w_2^{4-i} \right) x^2 z^2 + \left( \sum_{i=0}^8 d_i w_1^i w_2^{8-i} \right) x z^4 + \left( \sum_{i=0}^6 e_i w_1^i w_2^{6-i} \right) y z^3 \\ &+ \left( \sum_{i=0}^2 f_i w_1^i w_2^{2-i} \right) x y z + \left( \sum_{i=0}^{12} g_i w_1^i w_2^{12-i} \right) z^6 \\ &\equiv A_1 y^2 + A_2 x^3 + A_3 x^2 z^2 + A_4 x z^4 + A_5 y z^3 + A_6 x y z + A_7 z^6, \end{aligned}$$

where

$$A_i = A_i(w_1, w_2) \quad \text{for } i \geq 3.$$

Completing the square for  $y$  and then the cube for  $x$ , we get

$$\begin{aligned} p &= A_1 (y + B_1)^2 + A_2 \left( x + \frac{1}{3} B_2 z^2 \right)^3 + A_2 (B_3 - B_2^2) \left( x + \frac{1}{3} B_2 z^2 \right) z^4 \\ &+ \left( -\frac{A_2}{27} B_2^3 + \frac{A_2}{3} B_2 (B_3 - B_2) + B_4 \right) z^6 \\ &\equiv A_1 y'^2 + A_2 x'^3 - f(w_1, w_2) x' z^4 - g(w_1, w_2) z^6, \end{aligned}$$

$w_1^0 w_2^8 x z^4$	$w_1^0 w_2^{12} z^6$	$w_1^0 w_2^6 y z^3$	$w_1^0 w_2^4 x^2 z^2$	$w_1^0 w_2^2 x y z$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$w_1^8 w_2^0 x z^4$	$w_1^{12} w_2^0 z^6$	$w_1^6 w_2^0 y z^3$	$w_1^4 w_2^0 x^2 z^2$	$w_1^2 w_2^0 x y z$
$x^3$	$y^2$			

**Table 5.2:** Possible monomials for elliptic K3

where

$$B_1 = \frac{1}{2A_1} (A_5 z^3 + A_6 x z) \quad , \quad B_2 = \frac{1}{A_2} \left( A_3 - \frac{1}{4A_1} A_6^2 \right),$$

$$B_3 = \frac{1}{A_2} \left( A_4 - \frac{1}{2A_1} A_5 A_6 \right) \quad , \quad B_4 = A_7 - \frac{1}{4A_1} A_5^2$$

and

$$x' = x + \frac{1}{3} B_2 z^2 \quad , \quad y' = y + B_1.$$

Rescaling  $y'$  and  $x'$  and writing  $y$  and  $x$  respectively, we finally get the Weierstraß polynomial

$$p = y^2 - x^3 - f(w_1, w_2) x z^4 - g(w_1, w_2) z^6,$$

where  $f(w_1, w_2)$  and  $g(w_1, w_2)$  are homogeneous polynomials of degree 8 and 12. This nicely fits the general discussion of the elliptic fibration. The canonical bundle of  $\mathbb{P}^1$  is  $\mathcal{O}_{\mathbb{P}^1}(-2)$ , thus  $L = \mathcal{O}_{\mathbb{P}^1}(2)$ . Since the polynomial  $f$  and  $g$  are of degree 8 and 12 respectively, we have

$$f \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)) = \Gamma(\mathbb{P}^1, L^4) \quad \text{and} \quad g \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12)) = \Gamma(\mathbb{P}^1, L^6).$$

This is exactly the compactification described in [Vaf96]. The discriminant  $\Delta = 4f^3 + 27g^2 \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(24))$  is a polynomial of degree 24. This means that we have 24 7-branes.

### 5.3 The Orientifold Limit

We have seen that the F-theory describes a consistent 7-brane configuration of type IIB string theory. In this section we describe how one can obtain the type IIB orientifold compactification from the F-theory. We only consider a  $\mathbb{Z}_2$ -orientifold.

We start with an elliptic Calabi-Yau 4-fold  $Z$ . Let  $M$  be the base manifold and  $f \in \Gamma(M, L^4)$  and  $g \in \Gamma(M, L^6)$  the sections. We now describe the so-called **weak coupling limit** [Sen97, Sen98]. We rewrite

$$f = -3h^2 + C\eta,$$

$$g = -2h^3 + Ch\eta + C^2\chi,$$

where  $C$  is a constant and

$$h \in \Gamma(M, L^2) \quad , \quad \eta \in \Gamma(M, L^4) \quad \text{and} \quad \chi \in \Gamma(M, L^6).$$

Thus

$$\Delta = C^2 \left[ \eta^2 (4C\eta - 9h^2) + 54h (C\eta - 2h^2) \chi + 27C^2 \chi^2 \right].$$

We now take the limit

$$\begin{aligned} C &\rightarrow 0 \\ \Rightarrow \Delta &\approx C^2(-9h^2)(\eta^2 + 12h\chi). \end{aligned}$$

One can show that in this limit the string coupling constant is small in most of the base manifold  $M$  except, where  $|h| \sim |C|^{1/2}$ , thus justifying the name. The zeroes of  $\Delta$  are at

$$h^2 = 0 \quad \text{and} \quad \eta^2 + 12h\chi = 0.$$

By computing the monodromy around zeroes of  $\eta^2 + 12h\chi$ , we see that these zeroes represent D7-branes. Sen also calculates the monodromy around  $h^2 = 0$  and shows the zero locus of  $h$  corresponds to orientifold 7-planes. In his calculation Sen shows that the orientifold 7-planes are formed by two 7-branes in the weak coupling limit. In other words, the orientifold 7-planes split into two 7-branes if we go away from the weak coupling limit.

We can give the original Calabi-Yau threefold which would give the above orientifold. Let  $(u, \xi)$  denote the coordinate of  $L$ , where  $u$  denotes the coordinates of  $M$ . Let  $X$  denote the submanifold in the total space of the bundle  $L$  defined by the equation

$$\xi^2 = h.$$

This manifold has a  $\mathbb{Z}_2$ -isometry since

$$\sigma : \xi \longmapsto -\xi$$

does not change  $X$ . Let  $D$  denote the zero locus of  $h \in \Gamma(M, L^2)$ . The fixed point set of  $\sigma$  is precisely  $D$  which corresponds to the orientifold 7-planes. Thus  $X$  is the double cover of  $M$  branched over  $D$ .

From type IIB string theory point of view,  $X$  should be a Calabi-Yau manifold. Let us find out when this is the case. For a general double cover  $X$  of  $M$  branched over a zero locus  $D$  defined by a section of a line bundle  $L^2$  we have the formula [GH88, Prop. 2]

$$KX = \pi^*(KM \otimes L),$$

where  $\pi$  is the projection map from  $X$  to  $M$ . If we demand that  $X$  is Calabi-Yau, we obtain

$$c_1(KM) = -c_1(L) \Rightarrow c_1(M) = c_1(L)$$

which coincides with (5.3). Thus if  $Z$  is Calabi-Yau, then  $X$  is Calabi-Yau, too.

## 5.4 The Brane Moduli

The stabilization of 7-branes is a very important topic for semirealistic models. We have seen in § 5.1 that the brane moduli are encoded in the coefficients of the Weierstraß polynomial, i.e. the brane moduli correspond to polynomial deformations of the Calabi-Yau hypersurface. In this section we show the connection between the polynomial deformations and the complex structure deformations of the Calabi-Yau fourfolds using the techniques described in § 4.2 and the Kodaira-Spencer theory of complex structure deformations [KS58a, KS58b] or [Kod86]. This was done for hypersurfaces in  $\mathbb{P}^n$  in [KS58b].

The infinitesimal deformation of the complex structure of a general complex manifold is described by the sheaf cohomology group  $H^1(M, TM)$ . In general, there are **obstructions** such that not all elements

of  $H^1(M, TM)$  can be used to deform the complex structure. However, for Calabi-Yau manifolds the **Bogomolov-Tian-Todorov theorem** [Tia86] guarantees that every element of  $H^1(T, TM)$  corresponds to a deformation.

Calabi-Yau  $n$ -folds have a holomorphic volume form  $\Omega$  which can be used to define an isomorphism [Huy04, p.261] or [Joy00, p.142]

$$\eta : \wedge^p TM \xrightarrow{\cong} \wedge^{n-p} T^*M .$$

We have the fine resolution (4.5) for the sheaf  $\mathcal{O}_M(TM \otimes \wedge^p T^*M)$

$$0 \longrightarrow \mathcal{O}_M(TM \otimes \wedge^p T^*M) \longrightarrow \mathcal{A}^{p,0}(TM) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(TM) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2}(TM) \xrightarrow{\bar{\partial}} \dots .$$

With the isomorphism  $\eta$  we can replace  $TM$  in the resolution

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_M(\wedge^{n-1} T^*M \otimes \wedge^p T^*M) & \longrightarrow & \mathcal{A}^{p,0}(\wedge^{n-1} T^*M) & & \\ & & & & \searrow \bar{\partial} & & \\ & & \mathcal{A}^{p,1}(\wedge^{n-1} T^*M) & \xrightarrow{\bar{\partial}} & \mathcal{A}^{p,2}(\wedge^{n-1} T^*M) & \xrightarrow{\bar{\partial}} & \dots . \end{array}$$

Setting  $p = 0$ , we get

$$0 \longrightarrow \mathcal{O}_M(\wedge^{n-1} T^*M) \longrightarrow \mathcal{A}^{n-1,0}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{n-1,1}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{n-1,2}(M) \xrightarrow{\bar{\partial}} \dots \quad (5.7)$$

since  $\mathcal{A}^{0,q}(\wedge^p T^*M) = \mathcal{A}^{p,q}(M)$ . The sequence (5.7) is just the Dolbeault sequence. Thus we get

$$H^m(M, TM) \cong H^{n-1,m}(M). \quad (5.8)$$

The complex structure moduli of Calabi-Yau  $n$ -folds are described by the Dolbeault cohomology group  $H^{n-1,1}(M)$ .

Let  $X$  denote the ambient toric Fano variety of complex dimension 5 constructed from  $\Sigma(\Delta^*)$  and  $Z$  the Calabi-Yau hypersurface in  $X$ . Following Kodaira and Spencer [KS58a, (12.14)], we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_Z(TZ) \xrightarrow{l} \mathcal{O}_Z(TX|_Z) \xrightarrow{\pi} \mathcal{O}_Z([Z]|_Z) \longrightarrow 0 .$$

Note that we obtain this sequence applying  $\mathcal{O}_Z(\cdot)$  to the normal bundle sequence (2.2). As usual, we obtain a long exact sequence of cohomology groups from a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(Z, TZ) & \xrightarrow{l_*} & H^0(Z, TX|_Z) & \xrightarrow{\pi_*} & H^0(Z, [Z]|_Z) \\ & & & & \searrow \delta^0 & & \\ & & H^1(Z, TZ) & \xrightarrow{l_*} & H^1(Z, TX|_Z) & \xrightarrow{\pi_*} & H^1(Z, [Z]|_Z) \longrightarrow \dots . \end{array}$$

Unfortunately, the connecting homomorphism  $\delta^0$  is in general not surjective. As we will see, this means that there are complex structure moduli which do not stem from the polynomial deformations.

The aim from now on is to determine  $H^0(Z, TZ)$ ,  $H^0(Z, TX|_Z)$ , and  $H^0(Z, [Z]|_Z)$  to see why polynomial deformations have something to do with complex structure moduli.

First we turn to  $H^0(Z, TZ)$ . We have already shown in (5.8) that  $H^0(Z, TZ) \cong H^{3,0}(Z)$ , i.e.  $H^0(Z, TZ) = 0$  since  $H^{3,0}(Z) = 0$  for Calabi-Yau fourfolds.

Using the first vertical exact sequence of [KS58a, (12.18)]

$$0 \longrightarrow H^0(X, TX) \longrightarrow H^0(Z, TX|_Z) \longrightarrow 0,$$

we get  $H^0(X, TX) \cong H^0(Z, TX|_Z)$ .

Now we try to understand  $H^0(Z, [Z]|_Z)$ . There is an isomorphism [Cox93, Prop. 1.1])

$$H^0(X, [Z]) \cong \left\{ \sum_{m \in \Delta \cap M} \left( a_m \prod_i z_i^{\langle m, v_i \rangle + 1} \right) \right\},$$

between global holomorphic sections of the line bundle  $[Z]$  and the polynomials built from the monomials  $\prod_i z_i^{\langle m, v_i \rangle + 1}$ . The second vertical exact sequence of [KS58a, (12.17)] has the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(X, [Z]) & \longrightarrow & H^0(Z, [Z]|_Z) & (5.9) \\ & & & & & & \swarrow & \\ & & & & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, [Z]) & \longrightarrow & H^1(Z, [Z]|_Z) & \longrightarrow & \cdots \end{array}$$

where  $\mathcal{O}_X$  denotes the structure sheaf of  $X$ . As it is shown in [Cox93, Prop. 1.2.(i)],

$$H^0(X, \mathcal{O}_X) \cong \mathbb{C}. \quad (5.10)$$

Using [Oda85, Cor. 2.8]

$$H^q(X, \mathcal{O}_X) = 0 \quad \forall q > 0,$$

we obtain

$$H^1(X, \mathcal{O}_X) = 0. \quad (5.11)$$

Plugging (5.10) and (5.11) into (5.9) gives us following short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(X, [Z]) \longrightarrow H^0(Z, [Z]|_Z) \longrightarrow 0.$$

Thus we have

$$H^0(Z, [Z]|_Z) = H^0(X, [Z])/\mathbb{C},$$

where we will momentarily explain the modding out by  $\mathbb{C}$  intuitively.

We put everything together and obtain the exact sequence

$$0 \longrightarrow H^0(X, TX) \xrightarrow{\pi_*} H^0(X, [Z])/\mathbb{C} \xrightarrow{\delta^0} H^1(Z, TZ). \quad (5.12)$$

Now we see why polynomial deformations correspond to at least a subset of the complex structure moduli. Because the sequence is exact the map  $\pi_*$  is injective, we can mod  $H^0(X, TX)$  out of  $H^0(X, [Z])/\mathbb{C}$ . If additionally the map  $\delta^0$  was surjective,  $H^1(Z, TZ)$  would have been isomorphic to the quotient of  $H^0(X, [Z])/\mathbb{C}$  by  $H^0(X, TX)$ . However, since it is not surjective, in general the quotient of the two cohomology groups only forms a part of the complex structure moduli. In contrast to the toric construction, for the ambient space  $\mathbb{P}^n$  with  $n \geq 3$  and the degree of the polynomial bigger than 3, all complex structure moduli come from the polynomial deformations [KS58b, Thm. 14.2].

The appearance of the three cohomology groups can be understood intuitively. If we want to count the number of possible polynomial deformations, we do the following: we first count the number of monomials. This number corresponds to  $\dim H^0(X, [Z])$ . Since the zero locus, i.e. the hypersurface, does not change if the polynomial is multiplied by a global factor, we subtract 1. This is the intuitive reason why we mod out  $\mathbb{C}$  from  $H^0(X, [Z])$ . Then we subtract the number of coordinate transformations which leaves the hypersurface invariant, i.e. the number of infinitesimal automorphisms which is the Lie algebra of the automorphism group  $\text{Aut}(X)$  of the ambient space  $X$ . Since  $H^0(X, TX) \cong \text{Lie}(\text{Aut}(X))$  [Oda85, p.135], we also see this in the final exact sequence (5.12). Thus we obtain the exact sequence

$$0 \longrightarrow \text{Lie}(\text{Aut}(X)) \longrightarrow \left\{ \sum_{m \in \Delta \cap M} a_m \prod_i z_i^{\langle m, v_i \rangle + 1} \right\} / \mathbb{C} \longrightarrow H^{3,1}(Z).$$

This exact sequence and the formula for the Hodge number (3.10)

$$h^{n-1,1}(Z) = l(\Delta) - 1 - n - \sum_{\text{codim} \Theta=1} l^*(\Theta) + \sum_{\text{codim} \Theta=2} l^*(\Theta) \cdot l^*(\Theta^*)$$

give the number of the complex structure moduli which do not stem from the polynomial deformations. The dimension of  $\text{Lie}(\text{Aut}(X))$  is [CK99, Prop. 3.6.2]

$$\dim(\text{Lie}(\text{Aut}(X))) = n + \sum_{\text{codim} \Theta=1} l^*(\Theta).$$

The number of possible monomials is obviously equal to  $l(\Delta)$ . Thus the number of the moduli which do not correspond to the polynomial deformations is

$$\sum_{\text{codim} \Theta=2} l^*(\Theta) \cdot l^*(\Theta^*).$$

The  $-1$  in the formula for the Hodge number again corresponds to the modding out of  $\mathbb{C}$ .

Since we now know that brane moduli form a part of the complex structure moduli of the fourfold, we “just” have to stabilize two kinds of moduli: the complex structure moduli and the Kähler moduli. This is a big advantage over type IIB model building. There, the brane moduli constitute an additional class of moduli. In the F-theory, if we are able to fix the complex structure moduli, the brane moduli are fixed automatically.

## 5.5 Gukov-Vafa-Witten Superpotential

As we have shown in the previous section, we have to stabilize the complex structure moduli of the fourfolds used to compactify the F-theory in order to stabilize the brane moduli. There is a perturbative superpotential called **Gukov-Vafa-Witten superpotential** [GVW00]. It involves a 4-form flux  $G$  and has following form for the fourfold  $Z$

$$W = \frac{1}{2\pi} \int_Z \Omega \wedge G,$$

where  $\Omega$  is the holomorphic 4-form of  $Z$ . It can be shown that  $G$  is a  $(2, 2)$ -form [GVW00, § 2.4]. Thus after we have chosen a flux  $G \in H^4(Z, \mathbb{Z})$ , the complex structure, represented by the holomorphic 4-form  $\Omega$ , is adjusted such that  $G$  is of degree  $(2, 2)$  with respect to the adjusted  $\Omega$ . We see that the Gukov-Vafa-Witten superpotential generically fixes the complex structure of  $Z$ , thereby stabilizing also the 7-brane moduli.

# Heterotic Orbifolds

*[...] I was initiated into the mysteries of distributions  
by being told by a graduate student that physicists had used them,  
but understood nothing important about them.  
Only an innovative and brilliant mathematician  
like the idolized Laurent Schwartz could make sense of the physicists' nonsense.*  
Karen Uhlenbeck, [A<sup>+</sup>94]

There is one very popular way of building models which uses the heterotic toroidal orbifolds. The aim of this chapter is to provide toric geometry techniques to study such models, namely the resolution of non-compact Calabi-Yau orbifolds. Other references treating orbifolds in toric geometry context are [MOP87, Asp94, LRSS06]. Additionally, we give one proposal how a sensible intersection theory for non-compact manifolds can be defined.

## 6.1 Why Toric Geometry and Resolutions?

Heterotic orbifold models are very attractive since almost everything is calculable, e.g. the full spectrum and various couplings, because heterotic string theory on orbifolds corresponds to a sum of free conformal field theory. Additionally, using orbifolds like  $\mathbb{C}^3/\mathbb{Z}_{6-II}$ , it is possible to build models close to the minimal supersymmetric standard model [BHLR05, FNVW04, KRZ05, L<sup>+</sup>07]. However, it is often the case that these models are not stable due to the FI-terms [DSW87, DIS87] and it is believed that the theory is driven to a point in the field space, where the compactification is smooth and one has a non-abelian gauge background.

The best way to study these models would be to directly blow up the compact toroidal orbifolds, but an explicit blowup is not yet known. Therefore we look at local models of these orbifolds which means that we only look at one patch containing one fixed point. These local models are described by  $\mathbb{C}^3/\mathbb{Z}_n$ , which are non-compact Calabi-Yau orbifolds.

The resolution of an orbifold is a continuous process and as such it cannot alter topological properties of the theory, e.g. the chiral spectrum. Additionally, in the large volume limit, i.e. when the sizes of the exceptional cycles are large, all the higher excitations can be neglected. This means that we can approximate the theory by the super Yang-Mills theory coupled to supergravity. In [GNTW07] it was shown by using this approximation on explicitly constructed blowups of  $\mathbb{C}^n/\mathbb{Z}_n$  with U(1) gauge background that their chiral spectra could be recovered. Since the blowup was explicit, including the metric, it was possible to calculate all the necessary numbers, like the integrals over various Chern classes of the tangent bundle. However, for more complicated orbifolds an explicit blowup is not yet known.

Using toric geometry techniques, one can still extract enough information to obtain the chiral spectra of heterotic models on the resolution. To obtain the spectra we need intersection numbers of divisors, also those of non-compact divisors. In this chapter we focus on toric geometry techniques, i.e. how to construct the orbifolds, how to resolve them, and how one can compute the intersection numbers. In addition, we propose that intersections of divisors in non-compact toric varieties can be defined.

## 6.2 Toric Geometry Description of Orbifolds

We first want to describe how one can obtain the fan for non-compact Calabi-Yau orbifolds of type<sup>1</sup>  $\mathbb{C}^3/\mathbb{Z}_n$ .

The fan of  $\mathbb{C}^3/\mathbb{Z}_n$  has one three-dimensional cone which is spanned by three vectors and all of its faces. If we would naively apply the construction outlined in § 3.1.2, we would obtain  $\mathbb{C}^3$  because there are no charges and the exceptional set is empty. If no charges are present, we determine  $G = N/N'$ , where  $N' = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3$ . Since  $N'$  is a sublattice of  $N$ ,  $G$  will be a finite discrete group. The toric variety is constructed as

$$\mathbb{C}^3/G. \quad (6.1)$$

We now want to determine the vectors. Let the generator  $\theta$  of  $\mathbb{Z}_n$  act on the coordinates of  $\mathbb{C}^3$  according to

$$\theta(z_1, z_2, z_3) = (\varepsilon z_1, \varepsilon^{n_1} z_2, \varepsilon^{n_2} z_3),$$

where  $\varepsilon = e^{2\pi i/n}$ . Due to  $N = 1$  supersymmetry in four dimensions  $n_1$  and  $n_2$  have to satisfy

$$1 + n_1 + n_2 \equiv 0 \pmod{n}.$$

We characterize this action by a vector of the form

$$\frac{1}{n}(1, n_1, n_2).$$

Without loss of generality, we have set the first component of this vector to  $1/n$ . If none of the components was  $1/n$ , we would not have the full  $\mathbb{Z}_n$ -action. Assume for example the following vector for  $\mathbb{C}^3/\mathbb{Z}_4$

$$\frac{1}{4}(2, 2, 4).$$

Obviously, we would obtain only a  $\mathbb{Z}_2$ -action from this vector. To determine the  $v_i$ , we solve the equations

$$\left(e^{2\pi i/n}\right)^{v_{1i}} \left(e^{2\pi i n_1/n}\right)^{v_{2i}} \left(e^{2\pi i n_2/n}\right)^{v_{3i}} = 1 \quad \text{for } i = 1, 2, 3 \quad (6.2)$$

$$\Rightarrow v_{1i} + n_1 v_{2i} + n_2 v_{3i} \equiv 0 \pmod{n} \quad \text{for } i = 1, 2, 3. \quad (6.3)$$

In addition to these equations, we demand that

$$|\det(v_1, v_2, v_3)| = n.$$

Note that one solution to (6.3) is easily obtained, namely

$$v_{i3} = 1 \quad \text{for } i = 1, 2, 3$$

---

<sup>1</sup>Orbifolds of type  $\mathbb{C}^2/\mathbb{Z}_n$  are treated exactly like the three-dimensional examples without the third coordinate.

since  $1 + n_1 + n_2 \equiv 0 \pmod n$ . The three vectors  $v_i$  determined this way describes the orbifold  $\mathbb{C}^3/\mathbb{Z}_n$ . See § A.1 for details why this is the case, e.g. how (6.1) and (6.2) come about.

Now we look at some examples. First, we look at two-dimensional examples. Consider the orbifold  $\mathbb{C}^2/\mathbb{Z}_n$ . We have the following  $\theta$ -action

$$\frac{1}{n}(1, -1).$$

We have to find one solution to

$$v_{11} - v_{21} \equiv 0 \pmod n$$

which is linearly independent to  $v_{11} = v_{21} = 1$ . We choose  $v_{12} = 0$  and  $v_{22} = n$ . The determinant condition is obviously satisfied. Thus the matrix  $(V)$

$$\begin{pmatrix} 1 & 0 \\ 1 & n \end{pmatrix}$$

describes the orbifold  $\mathbb{C}^2/\mathbb{Z}_n$ .

As another example we consider the orbifold  $\mathbb{C}^3/\mathbb{Z}_4$ . The orbifold action is as follows

$$\frac{1}{4}(1, 1, 2).$$

We have to solve the equation

$$v_{1i} + v_{2i} + 2v_{3i} \equiv 0 \pmod 4.$$

We obtain the matrix  $(V)$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

The orbifold actions and the corresponding matrices  $(V)$  for  $\mathbb{C}^2/\mathbb{Z}_3$ ,  $\mathbb{C}^3/\mathbb{Z}_4$ ,  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{C}^3/\mathbb{Z}_{6-I}$ , and  $\mathbb{C}^3/\mathbb{Z}_{6-II}$  are given in § A.5.

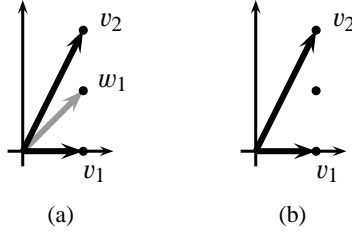
### 6.3 Resolution of Orbifolds

We want to resolve the orbifolds preserving the Calabi-Yau property. As described in § 3.2.2, this is called crepant resolution. It is quite easy to resolve a singular toric variety as already described in § 3.1.3. If the resolution should be crepant, we only take the integral vectors inside the fan satisfying the Calabi-Yau condition, i.e. vectors lying on the same hyperplane as the original vectors.

We first fix the conventions. We denote a general divisor, i.e. compact or non-compact, by  $D_i$ . To each vector  $v_i$  of the fan describing the orbifold we assign the coordinate  $z_i$  and denote the corresponding non-compact<sup>2</sup> divisor by  $\tilde{D}_i$ . We write  $w_j$  for the additional vectors in the resolved orbifold and  $x_j$  for the corresponding coordinate. The exceptional divisors are denoted by  $E_j$ . For convenience we write for

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<sup>2</sup>In § A.6 it is explained why these divisors are all non-compact.



**Figure 6.1:** Fans of  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$  and  $\mathbb{C}^2/\mathbb{Z}_2$

the extended  $(V | Q)$  matrix

$$\left( \begin{array}{cc|cccc|ccc} z_1 & \tilde{D}_1 & v_{11} & \cdots & v_{1k} & q_{11} & \cdots & q_{1\ell} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_n & \tilde{D}_n & v_{n1} & \cdots & v_{nk} & q_{n1} & \cdots & q_{n\ell} \\ x_1 & E_1 & w_{11} & \cdots & w_{1k} & q_{n+1,1} & \cdots & q_{n+1,\ell} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_m & E_m & w_{m1} & \cdots & w_{mk} & q_{n+m,1} & \cdots & q_{n+m,\ell} \end{array} \right),$$

where we have  $n$  vectors in the orbifold and  $m$  additional vectors in the resolved orbifold.

Let us look at a simple two-dimensional example, namely  $\mathbb{C}^2/\mathbb{Z}_2$ . Figure 6.1(b) shows the fan of this orbifold. The determinant of the matrix formed by  $v_1$  and  $v_2$  is 2. Thus we have a singular toric variety which is  $\mathbb{C}^2/\mathbb{Z}_2$ . We have to subdivide the cone spanned by  $v_1$  and  $v_2$  by adding integral vectors inside the cone such that the new vectors satisfy the Calabi-Yau condition and that each cone after the subdivision is spanned by vectors which constitute a  $\mathbb{Z}^2$ -basis. We add the generator  $w_1 = (1, 1)$  which is the only additional integral vector satisfying both requirements. Obviously this vector satisfies the Calabi-Yau condition, i.e.  $w_1$  lies on the same hyperplane, which is in this case just a line, as  $v_1$  and  $v_2$ . Now the subdivided fan has two two-dimensional cones and yields a smooth toric variety which is the resolved  $\mathbb{C}^2/\mathbb{Z}_2$  since  $\{v_1, w_1\}$  and  $\{v_2, w_1\}$  both form  $\mathbb{Z}^2$ -basis of  $\mathbb{Z}^2$ . The  $(V | Q)$  matrix is

$$\left( \begin{array}{cc|cc|c} z_1 & \tilde{D}_1 & 1 & 0 & 1 \\ z_2 & \tilde{D}_2 & 1 & 2 & 1 \\ x_1 & E_1 & 1 & 1 & -2 \end{array} \right).$$

The resolution is

$$\frac{\mathbb{C}^3 - \{z_1 = z_2 = 0\}}{(\mathbb{C}^*)}, \quad \text{where } (z_1, z_2, x_1) \sim (\lambda z_1, \lambda z_2, \lambda^{-2} x_1).$$

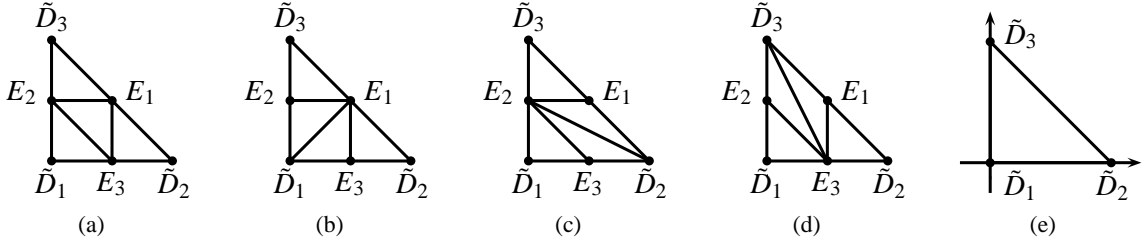
Figure 6.1(a) shows the fan of  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ .

In general the  $(V | Q)$  matrix of resolutions of orbifolds  $\mathbb{C}^n/\mathbb{Z}_n$  is

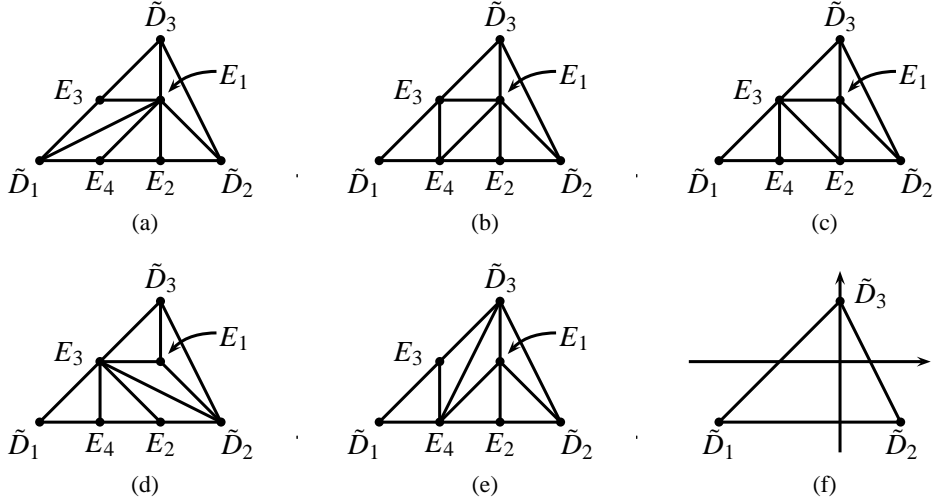
$$\left( \begin{array}{cc|cccc|c} z_1 & \tilde{D}_1 & 1 & \cdots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ z_{n-1} & \tilde{D}_{n-1} & 0 & \cdots & 1 & 1 & 1 \\ z_n & \tilde{D}_n & -1 & \cdots & -1 & 1 & 1 \\ x_1 & E_1 & 0 & \cdots & 0 & 1 & -n \end{array} \right) = \left( \begin{array}{cc|ccc|cc} z_1 & \tilde{D}_1 & & & & 1 & 1 \\ \vdots & \vdots & & & & \vdots & 1 \\ z_{n-1} & \tilde{D}_{n-1} & & \mathbf{1}_{n-1} & & 1 & 1 \\ z_n & \tilde{D}_n & -1 & \cdots & -1 & 1 & 1 \\ x_1 & E_1 & 0 & \cdots & 0 & 1 & -n \end{array} \right), \quad (6.4)$$

where  $\mathbf{1}_{n-1}$  denotes the  $(n-1) \times (n-1)$  identity matrix. The fan  $\Sigma$  consists of all cones of the form

$$\text{Cone}(w_1, v_{i_1}, \dots, v_{i_{n-1}}) \quad \forall i_k \neq i_\ell \quad \text{for } k \neq \ell. \quad (6.5)$$



**Figure 6.2:** The four different resolutions of  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$  and the fan of  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$



**Figure 6.3:** The five different resolutions of  $\mathbb{C}^3/\mathbb{Z}_{6-II}$  and the fan of  $\mathbb{C}^3/\mathbb{Z}_{6-II}$

and all faces thereof. The  $\mathbb{C}^*$ -action can be readily read off

$$(z_1, \dots, z_n, x_1) \sim (\lambda z_1, \dots, \lambda z_n, \lambda^{-n} x_1). \quad (6.6)$$

The exceptional set is as follows

$$Z(\Sigma) = \{z_1 = \dots = z_n = 0\}. \quad (6.7)$$

As a three-dimensional example, we consider the orbifold  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ . The  $(V | Q)$  matrix is given in (A.13). Since all vectors lie on a hyperplane, we can draw the fan projected to two dimensions which contains the same information as the three-dimensional diagram. To specify a fan with these additional vectors, we have to choose cones as mentioned in § 3.1.1. This choice of cones we also call triangulation because in the projected diagram of the fan this looks as if we would triangulate a surface bounded by the original vectors. The only difference between two different resolutions is the exceptional set since the exceptional set encodes information about the cones in the fan. Furthermore, we write  $\tilde{D}_i$  and  $E_j$  instead of  $v_i$  and  $w_j$  since we have the correspondence between the vectors and the divisors. Figure 6.2 shows the triangulated fans of different resolutions of  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ . Note that we obtain for example the triangulation 6.2(a) from 6.2(b), 6.2(c), and 6.2(d) through a flop and vice versa.

Similarly, figure 6.3 shows the five different resolutions of  $\mathbb{C}^3/\mathbb{Z}_{6-II}$  of which the  $(V | Q)$  matrix is given in (A.14).

## 6.4 Intersection Numbers of the Divisors

For model building in heterotic orbifolds we have to specify the gauge background to reduce the gauge group  $SO(32)$ . There is the standard embedding, where we use the curvature form of the Calabi-Yau manifold as the field strength of the gauge background, thus we get  $SU(n)$ -background for Calabi-Yau  $n$ -folds. The field strength of the gauge background should satisfy the Hermitian Yang-Mills equations, which is equivalent to Calabi-Yau condition in the case of the standard embedding. Another simple embedding is the  $U(1)$  gauge background, i.e. line bundles. In [GNTW07] an explicit  $U(1)$  gauge background satisfying the Hermitian Yang-Mills equations was constructed. Since divisors correspond to line bundles, we can use divisors as building blocks of the  $U(1)$  gauge background and use the intersection numbers to calculate integrals over the field strength. We now briefly describe why the intersection numbers of divisors, even numbers involving only non-compact divisors, are needed to compute the chiral spectra on the blowups.

Let now  $X$  be one of the non-compact resolved Calabi-Yau orbifolds. Let us assume we have calculated all intersection numbers of divisors  $\tilde{D}_i$  and  $E_j$ . The  $U(1)$  gauge background can be realized as a line bundle on  $X$ . Writing  $D_i \equiv c_1([D_i])$ , the field strength of the background can be expanded in terms of  $E_i$  with  $i = 1, \dots, m$  using the linear equivalences<sup>3</sup>. We embed the  $U(1)$ -background into the  $SO(32)$  gauge group as follows

$$\frac{\mathcal{F}_V}{2\pi} = \sum_{i=1}^m E_i \left( \sum_I V_{i,I} H_I \right),$$

where  $V_i$  are vectors with 16 components and the  $H_I$  are a basis of the Cartan subalgebra of  $\mathfrak{so}(32)$ . As a gauge background the field strength  $\mathcal{F}_V$  has to give integer numbers when integrated over compact cycles which is the quantization condition. This restricts the possible values of the  $V_i$ . If super Yang-Mills theories are coupled to supergravity, we have to fulfill the integrated Bianchi identity

$$\int_{C_4} (\text{tr } \mathcal{R}^2 - \text{tr}(i\mathcal{F}_V)^2) = 0,$$

where  $\mathcal{R}$  is the curvature form of the internal Calabi-Yau manifold and  $C_4$  is a compact 4-cycle of  $X$ . Additionally, we require that the Bianchi identity is satisfied for non-compact 4-cycles. The Bianchi identity puts further restriction on the  $V_i$  since we can compute the integral using the intersection numbers using the general formula for the Chern class of a toric variety. The total Chern class for an arbitrary smooth toric variety  $X$  [Ful93, p. 109]<sup>4</sup> has the form

$$c(X) = \prod_{i=1}^n (1 + D_i), \quad (6.8)$$

where we write  $D_i$  for  $c_1([D_i])$  and the sum runs over all vectors in the fan of  $X$ . In the case of resolved orbifolds the sum runs over all  $\tilde{D}_i$  and  $E_j$ . The  $i$ -th Chern class can be obtained by expanding the product. For example the first Chern class of  $X = \text{Res}(\mathbb{C}^n/\mathbb{Z}_n)$  is

$$c_1(X) = E_1 + \sum_{i=1}^n \tilde{D}_i = 0$$

<sup>3</sup>We expect that the Hermitian Yang-Mills equations can be solved by  $\mathcal{F}_V$ .

<sup>4</sup>There is a subtlety involved. The formula for the total Chern class is valid in the context of the Chow ring, i.e. the  $i$ -th Chern class has to be understood as an element of the corresponding Chow group  $A_i(X)$ . This can be problematic for non-compact manifold since  $A_0(X)$  can be trivial. If this is the case, it does not make sense to talk about intersection “numbers”. We ignore this fact and use the formula, also for non-compact toric varieties.

which confirms the Calabi-Yau condition of vanishing first Chern class. Also the second Chern class can be calculated easily

$$\frac{1}{2} \frac{\text{tr} \mathcal{R}^2}{(2\pi i)^2} = c_2(X) = E_1 \sum_i \tilde{D}_i + \sum_{i < j} \tilde{D}_i \cdot \tilde{D}_j = \frac{n+1}{2} E_1 \cdot \tilde{D}_1.$$

Thus once we have all intersection numbers of divisors, we can compute integrals over the Chern classes. To obtain the chiral spectrum we proceed as follows: we start from the anomaly polynomial  $I_{12} = I_{12}(\mathfrak{F}, \mathfrak{R})$  of the gaugino in 10 dimensions, where  $\mathfrak{F}$  and  $\mathfrak{R}$  are the full field strength and the full curvature form. We expand  $\mathfrak{F}$  and  $\mathfrak{R}$  in the background and the quantum part

$$\mathfrak{F} = \mathcal{F}_V + F \quad \text{and} \quad \mathfrak{R} = \mathcal{R} + R,$$

where  $\mathcal{F}_V$  and  $\mathcal{R}$  denote the background parts. Then we reduce  $I_{12}$  to four dimensions by integrating it over  $X$  to obtain  $I_6$  which has the form

$$I_6 \sim \left[ \text{tr} \left( \frac{1}{2} N_V (iF)^3 \right) - \frac{1}{8} \text{tr} \left( \frac{1}{2} N_V (iF) \right) \text{tr} R^2 \right],$$

where

$$N_V = \int_X \left[ \frac{1}{3!} \left( \frac{\mathcal{F}_V}{2\pi} \right)^3 + \frac{1}{12} \left( -\frac{1}{2} \frac{\text{tr} \mathcal{R}^2}{(2\pi i)^2} \right) \left( \frac{\mathcal{F}_V}{2\pi} \right) \right]. \quad (6.9)$$

We determine the unbroken gauge groups of  $\text{SO}(32)$  and the various matter multiplets in four dimensions and their charges under the gauge background. The charge of a matter multiplet under the  $\text{U}(1)$  gauge background is the eigenvalue of  $H_V$  and the multiplicity operator  $N_V$  computes how much this multiplet contributes to the anomaly. Thus for each multiplet  $N_V$  gives the number of chiral matter. Since  $N_V$  involves  $\mathcal{F}_V^3$  and  $R^2$  and therefore contains also intersection numbers of non-compact divisors only, we want to compute these numbers in addition to intersections involving at least one compact divisors.

We now describe how to obtain the intersection numbers. Let us first consider the case of a smooth compact toric variety. Let  $v_i$  for  $i = 1, \dots, n$  be the vectors in the fan  $\Sigma$  and  $D_i$  the corresponding divisors. To each vector in  $\Sigma$  we assign a homogeneous coordinate  $z_i$ . As explained in § 3.1.4, setting  $z_i = 0$  gives  $D_i$ . The linear equivalences are given by the vectors, i.e.

$$\sum_{i=1}^n v_{ij} D_i \sim 0.$$

Some of the intersections can be easily determined [Oda85, p. 80]. Let  $v_i, v_j$ , and  $v_k$  be the vectors corresponding to the divisors  $D_i, D_j$  and  $D_k$ . Then for  $i, j$  and  $k$  all different

$$D_i \cdot D_j \cdot D_k = 1$$

if  $v_i, v_j$  and  $v_k$  span a cone in  $\Sigma$ , and  $D_i \cdot D_j \cdot D_k = 0$  otherwise. We also have  $D_{i_1} \cdots D_{i_k} = 0$  if any subset of the involved divisors does not form a cone. For example, if  $\{z_1 = z_2 = 0\} \subset Z(\Sigma)$ , then for all  $i$  the intersection numbers of the form  $D_1 \cdot D_2 \cdot D_i$  will be equal to zero since the intersection  $D_1 \cdot D_2$  is defined by setting  $z_1$  and  $z_2$  to zero and this point set is excluded from the toric variety. Using these ‘‘initial’’ intersection numbers and linear equivalences we can determine all other intersection numbers including the self intersection numbers<sup>5</sup> like  $D_i \cdot D_i \cdot D_j \equiv D_i^2 \cdot D_j$  or  $D_i \cdot D_i \cdot D_i \equiv D_i^3$ .

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<sup>5</sup>The triple self intersection number of a divisor can be computed in a different way. See [Ful93, p.111].

Now we consider the non-compact case. We apply the results for the smooth compact toric varieties with one natural modification: when we determine the unit intersection numbers, we require that at least one compact divisor or one compact curve is involved. The points inside the projected fan correspond to compact divisors. The divisors associated to the boundary points, i.e. those on the triangle formed by  $v_1, v_2$ , and  $v_3$ , are non-compact. Like the correspondence between vectors, i.e. points in the projected fan, and divisors, there is a correspondence between two-dimensional cones, i.e. lines in the projected fan, and complex curves. Analogously to compact divisors, inner lines in the projected fan corresponds to compact complex curves and the lines on the outer triangle to non-compact complex curves. It is explained in § A.6 why this is the case.

As always we first consider a two-dimensional example:  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ . The exceptional divisor  $E_1$  is compact. The linear equivalences are

$$\tilde{D}_1 + \tilde{D}_2 + E_1 \sim E_1 + 2\tilde{D}_2 \sim 0. \quad (6.10)$$

Furthermore, we have the unit intersections

$$\tilde{D}_1 \cdot E_1 = \tilde{D}_2 \cdot E_1 = 1. \quad (6.11)$$

Combining (6.10) and (6.11), we obtain

$$\tilde{D}_1 \cdot \tilde{D}_2 = -\frac{1}{2}, \quad E_1^2 = -2 \quad \text{and} \quad \tilde{D}_i^2 = -\frac{1}{2} \quad \text{for} \quad i = 1, 2. \quad (6.12)$$

Two questions immediately arise: First of all, what does a fractional intersection number mean in a smooth manifold? Second, from the exceptional set we expect

$$\tilde{D}_1 \cdot \tilde{D}_2 = 0$$

since  $\{z_1 = z_2 = 0\}$  is contained in the exceptional set.

To see the problem more clearly, let us step back and see what linear equivalences mean and what we are calculating. Let therefore  $D_1$  be a compact and  $D_2$  and  $D_3$  non-compact divisors of a smooth non-compact complex two-dimensional manifold  $M$ . Furthermore, let us assume that  $D_1 \cdot D_2 = 1$  and that we have the linear equivalence

$$D_1 \sim D_3.$$

This is equivalent to

$$D_1 - D_3 = \delta V,$$

where  $\delta V$  is a boundary or equivalently a principal divisor. We obtain from  $D_1 \cdot D_2 = 1$

$$D_3 \cdot D_2 + \delta V \cdot D_2 = 1 \Leftrightarrow D_3 \cdot D_2 = 1 - \delta V \cdot D_2.$$

If we write the intersection  $\delta V \cdot D_2$  in terms of the corresponding  $(1, 1)$ -forms, i.e. the first Chern classes of the associated line bundles, we have

$$\delta V \cdot D_2 = \int_M c_1([\delta V]) \wedge c_1([D_2]) = \int_{D_2} c_1([\delta V]).$$

Since  $\delta V$  is a boundary, the corresponding first Chern class is exact. Thus

$$\delta V \cdot D_2 = \int_{D_2} d\phi$$

for some 1-form  $\phi$ . If  $D_2$  were compact, then the integral would vanish employing the Stokes theorem. However, because  $D_2$  is not compact, it does not vanish in general. As we calculated the intersection numbers (6.12), we neglected the contribution  $\delta V \cdot D_2$ . Thus fractional numbers can appear. This can be also understood from the global point of view. The non-compact resolved orbifold represents one local patch of a compact resolved orbifold. The boundary contributions will add up once we glue appropriate number of local patches together. Let us look at one example,  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_3)$  which is the local patch of  $T^4/\mathbb{Z}_3$ . For compact manifolds the Euler number is equal to the integral of the top Chern class. The second Chern class integrated over  $\mathbb{C}^2/\mathbb{Z}_3$  gives  $8/3$  neglecting boundary contributions. The resolution of the compact orbifold is the non-singular K3 whose Euler number is 24. The orbifold  $T^4/\mathbb{Z}_3$  has nine fixed points which means that we should glue nine of the local patches together. The integral of the second Chern class of  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_3)$  multiplied by the number of fixed points gives the Euler number of the K3.

We are also confronting the problem that in a non-compact manifold the intersection theory is not well defined. The intersection numbers of non-compact cycles are not topological in the sense that this number is not invariant under deformations of the cycles. One can deform the involved cycles such that they have no intersection at all. This is due to the fact that the intersection point can escape to infinity in a non-compact manifold. Let again  $D_1$  be a compact and  $D_2$  and  $D_3$  non-compact divisors. In addition, we assume  $D_1 \cdot D_2 = 1$  and

$$D_1 \sim aD_3 \quad a \in \mathbb{Z}. \quad (6.13)$$

To obtain the intersection between non-compact divisors  $D_2$  and  $D_3$ , we deform  $D_1$  using (6.13) to  $aD_3$ . By doing so, we forbid that the intersection point does escape to infinity which is the boundary of the non-compact manifold. We define the intersection between non-compact divisors imposing this restriction to the deformations. This means that  $aD_3 \cdot D_2 = 1$ . Thus  $D_3 \cdot D_2$  is fractional. To summarize, the intersection theory we consider neglects the boundary and is in that sense ‘‘local’’. This means one should understand the intersection theory we are considering as giving ‘‘local’’ information for one patch of the resolved compact orbifold.

Having given arguments for intersection theory on non-compact manifolds, we now actually compute these numbers for various examples. We first consider  $\mathbb{C}^n/\mathbb{Z}_n$ . We have the following linear equivalences which can be read off from (6.4)

$$\sum_{i=1}^n \tilde{D}_i + E_1 \sim 0 \quad \text{and} \quad \tilde{D}_i \sim \tilde{D}_j \quad \forall i \neq j \Rightarrow n\tilde{D}_i + E_1 \sim 0 \quad \forall i.$$

The unit intersection numbers obtained using (6.5) are

$$E_1 \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-1}} = 1 \quad \forall i_k \neq i_\ell \quad \text{for} \quad k \neq \ell. \quad (6.14)$$

Using these relations, we obtain

$$E_1^n = (-n)^{n-1}.$$

We compare these intersection numbers with integrals obtained in the explicit blowups. Using (6.6) and (6.7) we easily see that

$$\begin{aligned} E \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_p} &\cong \mathbb{P}^{n-p-1}, \\ \Rightarrow E \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-2}} &\cong \mathbb{P}^1 \end{aligned}$$

which is a compact curve. Thus (6.14) gives

$$E \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-2}} \cdot \tilde{D}_{i_{n-1}} = \int_{E \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-2}}} \tilde{D}_{i_{n-1}} = \int_{\mathbb{P}^1} \tilde{D}_{i_{n-1}} = 1.$$

In [GNTW07] the field strength  $\mathcal{F}$  of the U(1) gauge background obtained by solving the Hermitian Yang-Mills equation has the same property, i.e.

$$\frac{\mathcal{F}}{2\pi} = \tilde{D}_i \quad \text{and} \quad \int_{E \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-2}}} \frac{\mathcal{F}}{2\pi} = 1.$$

Furthermore, we then have

$$\begin{aligned} \int_{E_1 \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-1-p}}} \left( \frac{i\mathcal{F}}{2\pi i} \right)^p &= -n \int_{\tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-p}}} \left( \frac{i\mathcal{F}}{2\pi i} \right)^p = 1, \\ \Rightarrow -n \tilde{D}_1 \cdots \tilde{D}_n &= 1, \\ \Rightarrow \tilde{D}_1 \cdots \tilde{D}_n &= -\frac{1}{n} \end{aligned}$$

which agree with the integrals computed in [GNTW07]. Using (6.8) we can evaluate the integral

$$\int_{E_1 \cdot \tilde{D}_{i_1} \cdots \tilde{D}_{i_{n-2}}} c_2(X) = \int_{\mathbb{P}^2} c_2(X) = n(n+1)$$

which also perfectly agree with the results of explicit blowups. In [GNTW07] it was checked that the multiplicities of chiral matter computed using these intersection numbers match those of the corresponding heterotic orbifolds.

Despite issues due to non-compactness we are confident that the intersection numbers between non-compact divisors can make sense. For the case of  $\mathbb{C}^n/\mathbb{Z}_n$  for  $n = 2, 3$  we have two cross checks: the explicit blowups and the heterotic orbifold models. Since we need certain amount of ‘‘initial’’ intersection numbers for the calculation, we are confronting the problem which numbers are given and which should be calculated from the others using the linear equivalences. We propose following rules for  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_n)$  and  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_n)$ :

1. Basic cones, i.e. cones spanned by adjacent vectors, within the triangulation have unit intersection.
2. Basic cones which do not exist in the triangulation have zero intersection number.
3. Intersection numbers of divisors whose corresponding vectors are linearly dependent, i.e. the vectors lie on a line in the projected fan, have zero intersection number.
4. All other intersection numbers should be calculated using the unit and zero intersection numbers and the linear equivalences.

As another example we calculate the intersection numbers of the resolution of  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$  in Figure 6.2(a). Using the  $(V | Q)$  matrix of the fan  $\Sigma$  of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)$  given in (A.12), we can read off the linear equivalences

$$2\tilde{D}_1 + E_2 + E_3 \sim 2\tilde{D}_2 + E_1 + E_3 \sim 2\tilde{D}_3 + E_1 + E_2 \sim 0.$$

The exceptional set  $Z(\Sigma)$  is as follows

$$\begin{aligned} Z(\Sigma) &= \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\} \cup \{z_1 = z_3 = 0\} \\ &\cup \{z_1 = x_1 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{z_3 = x_3 = 0\}. \end{aligned}$$

Using the third rule, we have the following zero intersection numbers

$$\tilde{D}_1 \cdot \tilde{D}_2 \cdot E_3 = \tilde{D}_2 \cdot \tilde{D}_3 \cdot E_1 = \tilde{D}_3 \cdot \tilde{D}_1 \cdot E_2 = 0$$

	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$	$E_1$	$E_2$	$E_3$
$E_1 \cdot E_2$	0	0	1	-1	-1	1
$E_1 \cdot E_3$	0	1	0	-1	1	-1
$E_2 \cdot E_3$	1	0	0	1	-1	-1
$\tilde{D}_1 \cdot E_1$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0
$E_1 \cdot E_2$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	0
$E_1 \cdot E_2$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1

**Table 6.1:** All intersection numbers of the (a) triangulation of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)$  except the triple self intersection numbers.

which are triangulation independent, i.e. these intersection numbers coincide in all other triangulations. From the basic cones of the triangulation we obtain the unit intersection numbers

$$\tilde{D}_1 \cdot E_2 \cdot E_3 = \tilde{D}_2 \cdot E_3 \cdot E_1 = \tilde{D}_3 \cdot E_1 \cdot E_2 = E_1 \cdot E_2 \cdot E_3 = 1.$$

Using the second rule, we obtain following additional zero intersection numbers

$$\begin{aligned} \tilde{D}_1 \cdot E_1 \cdot E_2 &= \tilde{D}_1 \cdot E_1 \cdot E_3 = \tilde{D}_2 \cdot E_1 \cdot E_2 = 0, \\ \tilde{D}_2 \cdot E_2 \cdot E_3 &= \tilde{D}_3 \cdot E_1 \cdot E_3 = \tilde{D}_3 \cdot E_2 \cdot E_3 = 0. \end{aligned}$$

Employing the linear equivalences, this is enough to calculate intersection numbers involving only the exceptional divisors. Once we have all intersection numbers of the  $E_i$ , we can easily obtain all other numbers involving the  $D_i$  using the linear equivalences. The intersection numbers between exceptional divisors are as follows:

$$E_i^3 = -E_i^2 \cdot E_{j \neq i} = E_1 \cdot E_2 \cdot E_3 = 1 \quad \text{for } i = 1, 2, 3.$$

Table 6.1 shows all other intersection numbers except the triple self-intersection numbers.

In [GNHT07] it was checked using the intersection numbers calculated as above and the multiplicity operator  $N_V$  (6.9) that the number of chiral matter of heterotic orbifolds  $\mathbb{C}^2/\mathbb{Z}_3$ ,  $\mathbb{C}^3/\mathbb{Z}_4$  and  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$  could be recovered and that the anomalies were absent, confirming again the validity of the intersection of non-compact divisors.

However, the rules we proposed do not work for more complicated orbifolds. For example, for  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_{6-II})$  we can compute all intersection numbers between  $E_i$  except two numbers. The  $(V | Q)$  matrix is given in (A.14). The linear equivalences of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_{6-II})$  are as follows

$$6\tilde{D}_1 + E_1 + 2E_2 + 3E_3 + 4E_4 \sim 3\tilde{D}_2 + E_1 + 2E_2 + E_4 \sim 2\tilde{D}_3 + E_1 + E_3 \sim 0.$$

The ‘‘initial’’ intersection numbers are shown in table 6.2. The zero intersection numbers using the rules above are

$$\tilde{D}_1 \cdot E_2 \cdot E_4 = \tilde{D}_1 \cdot E_2 \cdot \tilde{D}_2 = E_2 \cdot E_4 \cdot \tilde{D}_2 = \tilde{D}_1 \cdot \tilde{D}_3 \cdot E_3 = \tilde{D}_3 \cdot E_1 \cdot E_2 = \tilde{D}_1 \cdot E_4 \cdot \tilde{D}_2 = 0.$$

Table 6.3 shows the intersection numbers between exceptional divisors computed using the unit and zero intersection numbers and the linear equivalences. As one can see in the table, we cannot determine the intersection numbers  $E_3^3$  and  $E_2 \cdot E_3^2$ .

	(a)	(b)	(c)	(d)	(e)
$E_1 \cdot E_2 \cdot E_3$	0	0	1	0	0
$E_1 \cdot E_2 \cdot E_4$	1	1	0	0	1
$E_2 \cdot E_3 \cdot E_4$	0	0	1	1	0
$E_1 \cdot E_3 \cdot E_4$	0	1	0	0	0
$\tilde{D}_1 \cdot E_1 \cdot E_4$	1	0	0	0	0
$\tilde{D}_1 \cdot E_1 \cdot E_3$	1	0	0	0	0
$\tilde{D}_1 \cdot E_3 \cdot E_4$	0	1	1	1	1
$\tilde{D}_2 \cdot E_1 \cdot E_2$	1	1	1	0	1
$\tilde{D}_2 \cdot E_1 \cdot \tilde{D}_3$	1	1	1	1	1
$\tilde{D}_2 \cdot E_1 \cdot E_3$	0	0	0	1	0
$\tilde{D}_2 \cdot E_2 \cdot E_3$	0	0	0	1	0
$\tilde{D}_3 \cdot E_1 \cdot E_3$	1	1	1	1	0
$\tilde{D}_3 \cdot E_1 \cdot E_4$	0	0	0	0	1
$\tilde{D}_3 \cdot E_3 \cdot E_4$	0	0	0	0	1

**Table 6.2:** “Initial” intersection numbers of the five different resolutions of  $\mathbb{C}^3/\mathbb{Z}_{6-II}$

	(a)	(b)	(c)	(d)	(e)
$E_1 \cdot E_2 \cdot E_3$	0	0	1	0	0
$E_1 \cdot E_2 \cdot E_4$	1	1	0	0	1
$E_2 \cdot E_3 \cdot E_4$	0	0	1	1	0
$E_1 \cdot E_3 \cdot E_4$	0	1	0	0	0
$E_1^2 \cdot E_2$	0	0	-1	0	0
$E_1^2 \cdot E_3$	0	-1	-2	-3	0
$E_1^2 \cdot E_4$	0	-1	0	0	-2
$E_1 \cdot E_2^2$	-2	-2	-1	0	-2
$E_1 \cdot E_3^2$	-2	-1	0	1	0
$E_1 \cdot E_4^2$	-2	-1	0	0	0
$E_2^2 \cdot E_3$	0	0	-1	-2	0
$E_2^2 \cdot E_4$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$E_2 \cdot E_4^2$	0	0	-1	-1	0
$E_3 \cdot E_4^2$	0	-1	-2	-2	0
$E_3^2 \cdot E_4$	0	-1	0	0	-2
$E_1^3$	6	7	8	9	8
$E_2^3$	2	2	1	2	2
$E_4^3$	2	1	2	2	0
$3E_3^3 + 2E_2 \cdot E_3^2$	8	5	0	-1	8

**Table 6.3:** All intersection numbers between exceptional divisors in the five different resolutions of  $\mathbb{C}^3/\mathbb{Z}_{6-II}$



# Conclusion and Outlook

*In particular, Theorem 6.5 is ineffective in explaining [...].  
Nevertheless, it is obvious that Theorem 6.5 is of fundamental importance [...].  
This reflects a fundamental difference between mathematics and physics.  
In physics, if a theory is developed for the purpose of explaining certain phenomenon  
and fails to do so, then that theory must be useless.*  
Kunihiko Kodaira, [Kod86]

In this thesis we described two applications of toric geometry to string theory. We focused on two particular theories, namely F-theory and heterotic toroidal orbifold theories. The main task for both applications was the construction of Calabi-Yau manifolds, both compact and non-compact, and studies thereof.

In the first part we introduced necessary concepts from algebraic geometry, especially divisors of a complex manifold. We discussed the basics of toric geometry and properties of toric varieties. Then we described two different methods to construct Calabi-Yau manifolds: non-compact Calabi-Yau manifolds as toric varieties and compact manifolds as hypersurfaces in toric Fano varieties using V. Batyrev's method. Since sheaf cohomology is necessary to discuss the brane moduli of F-theory, we explored basic elements of sheaves and their cohomologies.

For F-theory compactifications and model building we described the construction of *compact elliptic* Calabi-Yau fourfolds. Following this, we analyzed the brane moduli of F-theory compactifications and showed that the brane moduli do not represent a separate class of moduli, but they are contained in the complex structure moduli of the fourfolds. In addition we briefly discussed the Gukov-Vafa-Witten superpotential which leads to stabilization of the complex structure moduli.

Our second application deals with techniques for heterotic toroidal orbifold models. To analyze these models, we described how to construct *non-compact* Calabi-Yau orbifolds of type  $\mathbb{C}^3/\mathbb{Z}_n$  and how to resolve them preserving the Calabi-Yau condition. Since intersection numbers between divisors are needed, we described how to compute these numbers. While doing so, we had to face the fact that our manifolds were non-compact and therefore the intersection theory is ill defined. We gave arguments that a sensible definition of intersection theory of non-compact manifolds is possible. Using linear equivalences, we obtained fractional intersection numbers between non-compact divisors which were due to neglecting boundary contributions. We proposed a set of rules to compute the intersection numbers. Using these rules we were able to obtain all intersection numbers for  $\mathbb{C}^n/\mathbb{Z}_n, \mathbb{C}^2/\mathbb{Z}_3, \mathbb{C}_3/\mathbb{Z}_4$  and  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ . For these models one could check that the chiral spectra calculated from these numbers match those of heterotic orbifold models. In other words, it seems that intersection theory of non-compact manifolds makes sense. However, these rules are not yet sufficient for more complicated orbifolds, e.g.  $\mathbb{C}^3/\mathbb{Z}_{6-II}$ .

This work can be extended in many natural ways. For the F-theory part, the next step would be explicit model building. It would be also very important to study the mechanism of the stabilization of the complex structure moduli. Therefore one should compute the periods to obtain the explicit form of the perturbative superpotential. It seems possible to fix all moduli for toroidal compactifications [DDF<sup>+</sup>05]. It would be very interesting to see whether it is possible to fix all moduli in more complicated compactifications.

To obtain a more consistent and complete picture in heterotic orbifold models, it would be crucial to have a better understanding of the nature of fractional intersection numbers and of the intersection theory of non-compact manifold. Additionally, we should find a consistent prescription to compute all intersection numbers of arbitrary orbifolds. One possible way could be using the  $\text{Spec}(\cdot)$  construction and the extra information coming from the “sub-resolutions” as one can see in the following example: the projected fan of the resolution (b) of  $\mathbb{C}^3/\mathbb{Z}_{6-II}$  contains the projected fan of the resolution (a) of  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ . Since we have already computed all intersection numbers of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)$ , it might be possible to use them for computation of the intersection numbers of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_{6-II})$ . On the other hand we should also study the global picture of the orbifolds, i.e. the resolved compact orbifold. We should understand how one can consistently glue the local models together [LRSS06].

Finally, we hope that we will soon be able to build explicit models which are natural and good enough, such that string theory can become *useful physical* theory, even in Kodaira’s sense.

# Appendix

## A.1 Homogeneous Coordinates Revisited

In this appendix we describe the homogeneous coordinate construction in more detail.

Let  $N$  be the lattice and  $M$  the dual lattice with  $\dim N = \dim M = n$ . We consider only fans whose vectors  $\{v_i\}$  generate  $N_{\mathbb{R}}$  which means that we have  $r \geq n$  vectors,  $n$  of them linearly independent. Let  $X$  be a toric variety constructed via the  $\text{Spec}(\cdot)$  construction<sup>1</sup> from a fan  $\Sigma$  consisting of  $\{v_i\}$ . We have an exact sequence [Ful93, p.63]

$$0 \longrightarrow M \xrightarrow{\alpha} \bigoplus_i \mathbb{Z} \cdot D_i \xrightarrow{\beta} A_{n-1}(X) \longrightarrow 0, \quad (\text{A.1})$$

where  $D_i$  are the divisors corresponding to  $v_i$ . The map  $\alpha$  is defined as follows

$$\alpha(m) = \text{div}(\chi^m) = \sum_i \langle m, v_i \rangle D_i \quad \text{for } m \in M,$$

where  $\chi^m$  denote the element of the character group of  $M$  [Ful93, p.37]. We can view  $\chi^m$  as a rational function on  $X$ . Consequently  $\langle m, v_i \rangle = \text{ord}_{D_i}(\chi^m)$ . The  $(n-1)$ -th Chow group of  $X$  denoted by  $A_{n-1}(X)$  is the set of all divisors of  $X$  with linear equivalences<sup>2</sup> modded out. As we can see from (A.1),

$$A_{n-1}(X) = \bigoplus_i \mathbb{Z} \cdot D_i \Big/ M.$$

The map  $\beta$  maps  $D$  to its class  $A = \beta(D)$  in  $A_{n-1}(X)$ .

Let  $\text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^*)$  denote the set of all homomorphisms from  $X$  to  $\mathbb{C}^*$  with the property

$$f \in \text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^*) \Rightarrow f(ax) = f(x)^a \quad \text{for } a \in \mathbb{Z} \quad \text{and } x \in X. \quad (\text{A.2})$$

We apply  $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)$  to the exact sequence (A.1)

$$1 \longrightarrow \text{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*) \xrightarrow{\tilde{\beta}} \text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right) \xrightarrow{\tilde{\alpha}} \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \longrightarrow 1. \quad (\text{A.3})$$

Let  $G$  denote  $\text{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*)$ . Furthermore, we have

$$\text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^n = T(N),$$

where the third equality is a definition [Cox05, § 1.2]. We call  $T(N)$  the **torus** of  $N$ .

<sup>1</sup>See for example [Oda85, Ful93, Cox05].

<sup>2</sup>In the context of the Chow groups one should say **rational equivalences**. For divisors however, rational equivalence is linear equivalence.

Let us determine  $G$ . Since (A.3) is exact,

$$\text{Ker } \tilde{\alpha} = \text{Im } \tilde{\beta} \cong G.$$

Let  $h \in \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$  and  $f \in \text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right)$ . Obviously,

$$\text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n \quad \text{and} \quad \text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right) \cong (\mathbb{C}^*)^r.$$

By choosing a basis for each of both spaces, we can write

$$h = (h_1, \dots, h_n) \quad \text{and} \quad f = (f_1, \dots, f_r).$$

Let  $m \in M$  and  $D = \sum_i a_i D_i \in \bigoplus_i \mathbb{Z} \cdot D_i$ , then we have

$$\begin{aligned} h(m) &= (h_1, \dots, h_n) \cdot (m_1, \dots, m_n) = \prod_i h_i^{\langle m, e_i \rangle} = \prod_i h_i^{m_i}, \\ f(D) &= (f_1, \dots, f_r) \cdot D = \prod_i f(D_i)^{a_i} = \prod_i f_i^{a_i}, \end{aligned}$$

where  $e_i$  are the standard basis of  $N$ . Obviously  $h$  and  $f$  defined this way are homomorphisms and satisfy (A.2). To determine  $G$ , we have to determine  $\text{Ker } \tilde{\alpha}$ . The map  $\tilde{\alpha}$  is defined as

$$\tilde{\alpha} : \text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right) \longrightarrow T(N)$$

and since  $\tilde{\alpha}(f) \in \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ , we define  $\tilde{\alpha}(f)$  as follows

$$\tilde{\alpha}(f)(m) = \prod_{i=1}^n \prod_{j=1}^r f_j^{\langle m, v_i \rangle} \quad \text{for } m \in M.$$

If we plug in the standard basis  $e_i^*$  of  $M$  in  $\tilde{\alpha}(f)(\cdot)$ , we obtain the representation of  $\tilde{\alpha}(f)$  in components, namely

$$\tilde{\alpha}(f) = \left( \prod_{j=1}^r f_j^{v_{j1}}, \dots, \prod_{j=1}^r f_j^{v_{jn}} \right).$$

Thus  $f = (f_1, \dots, f_r) \in G$  iff

$$\prod_{j=1}^r f_j^{v_{ji}} = 1 \quad \forall i.$$

To summarize,

$$G = \left\{ (f_1, \dots, f_r) \mid \prod_{j=1}^r f_j^{v_{ji}} = 1 \quad \text{for } i = 1, \dots, n \right\}.$$

From the exact sequence (A.3) we see that

$$T(N) = (\mathbb{C}^*)^r / G,$$

i.e.

$$\left( \prod_{j=1}^r f_j^{v_{j1}}, \dots, \prod_{j=1}^r f_j^{v_{jn}} \right)$$

can be seen as coordinates of  $T(N)$  and they are invariant under  $G$ .

An action of the group  $G$  on  $\mathbb{C}^r$  can be defined via

$$g \in G \Rightarrow g \cdot (z_1, \dots, z_r) = (g(D_1)z_1, \dots, g(D_r)z_r).$$

We can have two cases: first, the vectors  $\{v_i\}$  are linearly dependent. In this case it is easy to see that  $G$  corresponds to the  $(\mathbb{C}^*)^m$  action weighted by the charges: We first set

$$f_i = \prod_k \lambda_k^{q_{ik}},$$

then

$$1 = \prod_{i,k} \lambda_k^{q_{ik} v_{ij}} \Rightarrow \sum_i q_{i\ell} v_{ij} = 0 \quad \forall j, \ell.$$

The  $q_{i\ell}$  are precisely the charges defined in § 3.1.2.

Let  $\{v_i\}$  be linearly independent. Then we have precisely  $\dim N = n$  vectors in our fan since  $\Sigma(1)$  spans  $N_{\mathbb{R}}$ . We now show that

$$G \cong N/N',$$

where  $N' = \bigoplus_{i=1}^n \mathbb{Z} \cdot v_i$ . One can show that the following sequence

$$1 \longrightarrow N/N' \longrightarrow T(N') \longrightarrow T(N) \longrightarrow 1$$

is exact [Ful93, p.34]. Furthermore, using the following isomorphism

$$\bigoplus_i \mathbb{Z} \cdot D_i \cong N'$$

which is due to the one-to-one correspondence between divisors and vectors, we obtain

$$T(N') = \text{Hom}_{\mathbb{Z}}(N', \mathbb{C}^*) \cong \text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right).$$

Thus we have the commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & N/N' & \longrightarrow & T(N') & \longrightarrow & T(N) \longrightarrow 1 \\ \downarrow = & & \downarrow \phi & & \downarrow \cong & & \downarrow = \\ 1 & \longrightarrow & G & \longrightarrow & \text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right) & \longrightarrow & T(N) \longrightarrow 1 \end{array} \quad (\text{A.4})$$

First we show that the map  $\phi$  exists and makes the diagram commute. Consider the diagram:

$$\begin{array}{ccc} N/N' & & T(N') \\ \downarrow \phi & \searrow \psi & \\ G & \xrightarrow{\tilde{\beta}} & T(N') \end{array}$$

For simplicity we have identified  $T(N')$  and  $\text{Hom}_{\mathbb{Z}}\left(\bigoplus_i \mathbb{Z} \cdot D_i, \mathbb{C}^*\right)$ . Let  $w \in N/N'$ , then  $\psi(w) \in T(N')$ . Since  $\tilde{\beta}$  is injective, we can consider  $g = \tilde{\beta}^{-1}(\psi(w)) \in G$ . We set  $\phi(w) = g$ . By construction  $\phi$  makes the diagram commute. Using the same argument we can construct  $\phi'$ . We want to show that  $N/N'$  and  $G$  are isomorphic. To do this, we use the so-called “four lemma” [Mac95, p.14] which states: Assume we have a commutative diagram

$$\begin{array}{cccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 \end{array}$$

with two exact rows. If  $f_1$  and  $f_3$  are surjective and  $f_4$  is injective, then  $f_2$  is surjective. Since all vertical solid arrows in (A.4) are bijective, we can use the four lemma identifying the dashed arrow with  $f_2$ . Using the four lemma twice for  $\phi$  and  $\phi'$ , from the upper row to the lower row and vice versa, we obtain that

$$\phi : N/N' \longrightarrow G \quad \text{and} \quad \phi' : G \longrightarrow N/N'$$

are both surjective. The dashed arrow in (A.4) is thus an isomorphism. This means in particular that  $\text{ord}(G) = \text{ord}(N/N')$ , thus

$$|\det(v_1, \dots, v_n)| = \text{ord } G.$$

The quotient

$$\frac{\mathbb{C}^r - Z(\Sigma)}{G},$$

where  $Z(\Sigma)$  is the exceptional set defined in § 3.1.2, is naturally isomorphic to the toric variety constructed from  $\Sigma$  using the  $\text{Spec}(\cdot)$  construction [Cox93, Thm. 2.1].

## A.2 Linear Equivalences of Divisors

We have the following exact sequence for an arbitrary toric variety  $X$  [Cox05, (3.2)]

$$M \xrightarrow{\alpha} \bigoplus_i \mathbb{Z} \cdot D_i \xrightarrow{\beta} A_{n-1}(X) \longrightarrow 0,$$

where  $D_i$  are the divisors corresponding to  $v_i$ . Note that this sequence becomes a short exact sequence if the toric variety is compact. As already mentioned in § A.1, the map  $\alpha$  is defined to be

$$\alpha(m) = \text{div}(\chi^m) = \sum_i \langle m, v_i \rangle D_i \quad \text{for } m \in M,$$

i.e.<sup>3</sup>

$$A_{n-1}(X) = \bigoplus_i \mathbb{Z} \cdot D_i \Big/ \text{Im}(\alpha).$$

Thus the divisors which are mapped to the trivial class in  $A_{n-1}(X)$  has the form

$$\sum_i \langle m, v_i \rangle D_i \quad \forall m \in M.$$

If we take the standard basis vectors  $e_i^*$  for  $i = 1 \dots n$  of  $M$ , we obtain

$$0 \sim \sum_j \langle e_i^*, v_j \rangle D_j = \sum_j v_{ji} D_j \quad \text{for } i = 1 \dots n, \quad (\text{A.5})$$

where we write  $v_{ji}$  for the  $i$ -th component of the  $j$ -th vector. Furthermore, we can recast (A.5) to give another meaning to charges. Consider the  $(V \mid Q)$  matrix of the form

$$\left( \begin{array}{ccc|ccc} v_{11} & \cdots & v_{1n} & q_{11} & \cdots & q_{1\ell} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{k1} & \cdots & v_{kn} & q_{k1} & \cdots & q_{k\ell} \end{array} \right).$$

We have the relations

$$\sum_i q_{ij} v_{im} = 0 \quad \text{and} \quad \sum_i v_{im} D_i \sim 0 \quad \text{for } j = 1, \dots, \ell \quad \text{and} \quad m = 1, \dots, k.$$

---

<sup>3</sup>We mod out  $\text{Im}(\alpha)$  because  $\alpha$  is not injective. We cannot identify  $\text{Im}(\alpha)$  with  $M$ .

The components of the vectors  $v_{im}$  can be seen as coefficients of linear combinations of the rows of  $(Q)$  which sum up to zero. Thus we can interpret the rows of  $(Q)$  as *formal* coordinates of each divisor  $D_i$ . Let  $Q_i$  denote the  $i$ -th row of  $(Q)$ . Then the divisor  $D_i$  for example has  $Q_i$  as its coordinate. Since

$$\sum_i v_{ij} Q_i = 0,$$

we get

$$\sum_i v_{ij} D_i \sim 0.$$

This means that there are  $\ell$  linearly independent divisors and thus  $k - \ell$  independent linear equivalences. These equivalences can be found by considering the linear combinations of  $Q_i$  which sum up to zero.

### A.3 Polynomial defining Calabi-Yau Hypersurfaces

Let  $X$  be a toric Fano variety. We go through a reasoning why the polynomial  $p$  as defined in (3.8)

$$\sum_{m \in \Delta \cap M} \left( a_m \prod_i z_i^{\langle m, v_i \rangle + 1} \right)$$

is defining a Calabi-Yau variety, i.e. a section in the anticanonical bundle.

We have the following short exact sequence for compact toric varieties [Ful93, p.63]

$$0 \longrightarrow M \xrightarrow{\alpha} \bigoplus_i \mathbb{Z} \cdot D_i \xrightarrow{\beta} A_{n-1}(X) \longrightarrow 0,$$

where  $D_i$  are the divisors corresponding to  $v_i$ . The map  $\alpha$  is defined as follows

$$\alpha(m) = \operatorname{div}(\chi^m) = \sum_i \langle m, v_i \rangle D_i \quad \text{for } m \in M.$$

The map  $\beta$  maps  $D$  to its class  $A = \beta(D)$  in the  $(n - 1)$ -th Chow group  $A_{n-1}(X)$ .

For  $D = \sum_i d_i D_i$  the group of global sections  $H^0(X, [D])$  can be given as [Ful93, p.66]

$$H^0(X, [D]) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m,$$

where the polyhedron  $P_D$  is defined as

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, v_i \rangle \geq -d_i \forall i\}. \quad (\text{A.6})$$

Following [Cox93, § 1], let us define

$$S_{\beta(D)} = \bigoplus_{\beta(D')=A} \mathbb{C} \cdot z^{D'},$$

where  $D' = \sum_i b_i D_i$  and

$$z^{D'} = \prod_i z_i^{b_i}.$$

Since  $\operatorname{Ker} \beta = \operatorname{Im} \alpha$ ,

$$\prod_i z_i^{b_i}, \prod_i z_i^{c_i} \in S_{\alpha} \Rightarrow \exists m \in M : b_i = \langle m, v_i \rangle + c_i \forall i. \quad (\text{A.7})$$

We have an isomorphism [Cox93, Prop. 1.1]

$$\phi_D : S_{\beta(D)} \xrightarrow{\cong} H^0(X, [D]).$$

The isomorphism  $\phi_D$  is defined as follows: let  $D_m = \sum_i \langle m, v_i \rangle D_i$ , then

$$z^{D'} \in S_A \Rightarrow \exists m \in P_D \cap M : z^{D'} = z^{D_m+D} = \prod_i z_i^{\langle m, v_i \rangle + d_i}. \quad (\text{A.8})$$

Thus

$$S_{\beta(D)} = \left\{ \sum_{m \in P_D \cap M} \left( a_m \prod_i z_i^{\langle m, v_i \rangle + d_i} \right) \right\}.$$

We set

$$\phi_D(z^{D'}) = \phi_D(z^{D_m+D}) = \chi^m. \quad (\text{A.9})$$

Let us turn to the reflexive polyhedron  $\Delta$ . Now  $\Delta$  plays the role of  $P_D$ . Using  $(\Delta^*)^* = \Delta$ , we get

$$\Delta = \{x \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \ \forall y \in \Delta^*\} \in M_{\mathbb{R}}.$$

A glance at (A.6) shows that the corresponding divisor is

$$D = \sum_i D_i,$$

where the sum runs over all vectors generating the one-dimensional cones in the fan corresponding to  $\Delta^*$ , i.e. also vectors added by crepant desingularization. This divisor  $D$  is the anticanonical divisor of  $X$  [Ful93, p.85]. We now determine the corresponding  $S_{\beta(D)}$  which is isomorphic to the global sections of the line bundle  $[D] = [-K_X]$ , i.e. of the anticanonical bundle  $KX^{-1}$ . The monomials spanning  $S_{\beta(D)}$  have the form using (A.8) and (A.9)

$$z^{D'} \in S_{\beta(D)} \Rightarrow \prod_i z_i^{\langle m, v_i \rangle + 1} \quad \text{for } m \in \Delta \cap M.$$

The reason exponents being of the form  $\langle m, v_i \rangle + 1$  is that  $d_i = 1$  for all  $i$  which is the defining property of dual reflexive polyhedrons. Thus every global section of  $KX^{-1}$  has the form

$$\sum_{m \in \Delta \cap M} \left( a_m \prod_i z_i^{\langle m, v_i \rangle + 1} \right)$$

in homogeneous coordinates.

## A.4 Integral Points of Reflexive Polyhedrons

Here we give the integral points of reflexive polyhedrons for elliptic K3 studied in § 5.2.6.

The vertices of  $\Delta^*$ :

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The integral points of  $\Delta^*$ , i.e.  $\Delta^* \cap N$ :

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & -3 \\ -1 & -4 & -6 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{array} \right). \quad (\text{A.10})$$

The vertices of  $\Delta$ :

$$\begin{pmatrix} 11 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

The integral points:

$$\left( \begin{array}{ccc|ccc|ccc} -1 & 0 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 2 & 0 & -1 & 2 & -1 & -1 & 2 & 1 & -1 \\ 3 & 0 & -1 & 3 & -1 & -1 & 3 & 1 & -1 \\ 4 & 0 & -1 & 4 & -1 & -1 & -1 & -1 & 0 \\ 5 & 0 & -1 & 5 & -1 & -1 & 0 & -1 & 0 \\ 6 & 0 & -1 & 6 & -1 & -1 & 1 & -1 & 0 \\ 7 & 0 & -1 & 7 & -1 & -1 & 2 & -1 & 0 \\ -1 & 2 & -1 & 8 & -1 & -1 & 3 & -1 & 0 \\ -1 & -1 & 1 & 9 & -1 & -1 & 4 & -1 & 0 \\ -1 & 0 & 0 & 10 & -1 & -1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 11 & -1 & -1 & & & \\ 1 & 0 & 0 & & & & & & \end{array} \right), \quad (\text{A.11})$$

where we tried to group points with similar components together.

## A.5 The $(V | Q)$ Matrix of Orbifolds and their Resolutions

For notation see the beginning of § 6.2.

### A.5.1 $\mathbb{C}^2/\mathbb{Z}_3$

The orbifold action:

$$\theta(z_1, z_2) = (\varepsilon z_1, \varepsilon^{-1} z_2),$$

where  $\varepsilon = e^{2\pi i/3}$ .

Equations to solve:

$$v_{1i} - v_{2i} \equiv 0 \pmod{3}.$$

The  $(V | Q)$  matrix of  $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ :

$$\left( \begin{array}{cc|cc|cc} z_1 & \tilde{D}_1 & 1 & 0 & 1 & 0 \\ z_2 & \tilde{D}_2 & 1 & 3 & 0 & 1 \\ x_1 & E_1 & 1 & 1 & -2 & 1 \\ x_2 & E_2 & 1 & 2 & 1 & -2 \end{array} \right).$$

### A.5.2 $\mathbb{C}^3/\mathbb{Z}_4$

The orbifold action:

$$\theta(z_1, z_2, z_3) = (\varepsilon z_1, \varepsilon z_2, \varepsilon^2 z_3),$$

where  $\varepsilon = e^{2\pi i/4}$ .

Equation to solve:

$$v_{1i} + v_{2i} + 2v_{3i} \equiv 0 \pmod{4}.$$

The  $(V | Q)$  matrix of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_4)$ :

$$\left( \begin{array}{cc|ccc|cc} z_1 & \tilde{D}_1 & 2 & 0 & 1 & 1 & 0 \\ z_2 & \tilde{D}_2 & 0 & 2 & 1 & 1 & 0 \\ z_3 & \tilde{D}_3 & -1 & 1 & 1 & 0 & 1 \\ x_1 & E_1 & 0 & 1 & 1 & 0 & -2 \\ x_2 & E_2 & 1 & 1 & 1 & -2 & 1 \end{array} \right).$$

### A.5.3 $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$

The orbifold action:

$$\theta_1(z_1, z_2, z_3) = (\varepsilon z_1, z_2, \varepsilon z_3),$$

$$\theta_2(z_1, z_2, z_3) = (z_1, \varepsilon z_2, \varepsilon z_3),$$

$$\theta_1 \circ \theta_2(z_1, z_2, z_3) = (\varepsilon z_1, \varepsilon z_2, z_3),$$

where  $\varepsilon = e^{\pi i}$ .

Equations to solve:

$$v_{1i} + v_{3i} \equiv 0 \pmod{2},$$

$$v_{2i} + v_{3i} \equiv 0 \pmod{2},$$

$$v_{1i} + v_{2i} \equiv 0 \pmod{2}.$$

The  $(V | Q)$  matrix of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2)$ :

$$\left( \begin{array}{cc|ccc|ccc} z_1 & \tilde{D}_1 & 0 & 0 & 1 & 0 & 1 & 0 \\ z_2 & \tilde{D}_2 & 0 & 2 & 1 & 1 & 0 & 0 \\ z_3 & \tilde{D}_3 & 2 & 0 & 1 & 0 & 0 & 1 \\ x_1 & E_1 & 1 & 1 & 1 & -1 & 1 & -1 \\ x_2 & E_2 & 0 & 1 & 1 & -1 & -1 & 1 \\ x_3 & E_3 & 1 & 0 & 1 & 1 & -1 & -1 \end{array} \right). \tag{A.12}$$

### A.5.4 $\mathbb{C}^3/\mathbb{Z}_{6-l}$

The orbifold action:

$$\theta(z_1, z_2, z_3) = (\varepsilon z_1, \varepsilon z_2, \varepsilon^{-2} z_3),$$

where  $\varepsilon = e^{2\pi i/6}$ .

Equation to solve:

$$v_{1i} + v_{2i} - 2v_{3i} \equiv 0 \pmod{6}.$$

The  $(V | Q)$  matrix of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_{6-I})$ :

$$\left( \begin{array}{ccc|ccc} z_1 & \tilde{D}_1 & & 1 & -2 & 1 & 1 & 0 & 0 \\ z_2 & \tilde{D}_2 & & -1 & -2 & 1 & 1 & 0 & 0 \\ z_3 & \tilde{D}_3 & & 0 & 1 & 1 & 0 & 1 & 0 \\ x_1 & E_1 & & 0 & 0 & 1 & 0 & -2 & 1 \\ x_2 & E_2 & & 0 & -1 & 1 & 0 & 1 & -2 \\ x_3 & E_3 & & 0 & -2 & 1 & -2 & 0 & 1 \end{array} \right). \quad (\text{A.13})$$

### A.5.5 $\mathbb{C}^3/\mathbb{Z}_{6-II}$

The orbifold action:

$$\theta(z_1, z_2, z_3) = (\varepsilon z_1, \varepsilon^2 z_2, \varepsilon^3 z_3),$$

where  $\varepsilon = e^{2\pi i/6}$ .

The equation to solve:

$$v_{1i} + 2v_{2i} + 3v_{3i} \equiv 0 \pmod{6}.$$

The  $(V | Q)$  matrix of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_{6-II})$ :

$$\left( \begin{array}{ccc|cccc} z_1 & \tilde{D}_1 & & -2 & -1 & 1 & 1 & -1 & 1 & 0 \\ z_2 & \tilde{D}_2 & & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ z_3 & \tilde{D}_3 & & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ x_1 & E_1 & & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ x_2 & E_2 & & 0 & -1 & 1 & 0 & 0 & 1 & -2 \\ x_3 & E_3 & & -1 & 0 & 1 & -2 & 1 & 0 & 0 \\ x_4 & E_4 & & -1 & -1 & 1 & 0 & 1 & -2 & 1 \end{array} \right). \quad (\text{A.14})$$

## A.6 Compact Divisors and Compact Curves in Resolved Orbifolds

In this appendix we explain why divisors corresponding to a vector which is inside the projected fan is compact and likewise why the complex curves corresponding to the inner lines of the projected fan are compact.

Let  $X$  be an arbitrary  $n$ -dimensional toric variety constructed from a fan  $\Sigma$  in the lattice  $N_{\mathbb{R}}$ . The fan  $\Sigma$  in general contains  $n$  to zero-dimensional cones. We now describe the **star** of a cone [Ful93, § 3.1]. Let  $\sigma$  be a  $k$ -dimensional cone of  $\Sigma$ . Let  $N(\sigma)$  denote the lattice spanned by the vectors of  $\sigma$ , i.e. without loss of generality

$$\sigma = \text{Cone}(v_1, \dots, v_k) \Rightarrow N(\sigma) = \bigoplus_{i=1}^k \mathbb{Z} \cdot v_i.$$

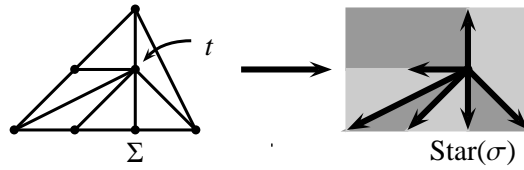
Furthermore, let  $p$  denote the projection map

$$p : N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}/N(\sigma)_{\mathbb{R}},$$

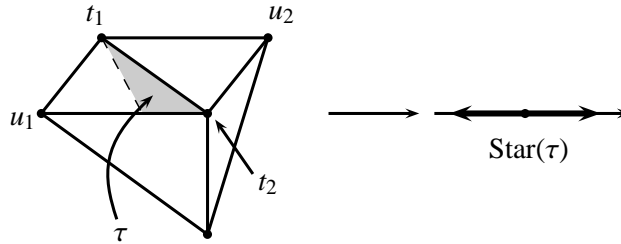
where  $N(\sigma)_{\mathbb{R}} = N(\sigma) \otimes \mathbb{R}$ . Then the star of  $\sigma$  is defined as follows

$$\text{Star}(\sigma) = \bigcup_{\sigma < \tau \in \Sigma} p(\tau),$$

where  $\sigma < \tau$  means that  $\sigma$  is a face of  $\tau$ . The star of a  $k$ -dimensional cone is a  $(n - k)$ -dimensional fan. The toric variety constructed from  $\text{Star}(\sigma)$  denoted by  $V(\sigma)$  is an  $(n - k)$ -dimensional subvariety of  $X$ . These  $(n - k)$ -dimensional subvarieties constructed from the stars generate the  $(n - k)$ -th Chow group



**Figure A.1:** Example of  $\text{Star}(\sigma)$  from a fan  $\Sigma$ . The one-dimensional cone  $\sigma$  is spanned by the vector  $t$ .



**Figure A.2:** Star of a two-dimensional cone  $\tau$

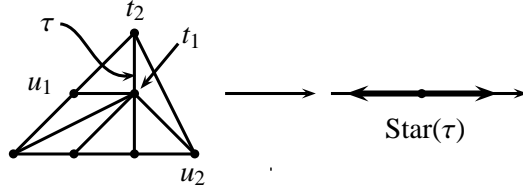
$A_{n-k}(X)$  [Ful93, § 5.1]. The  $(n - 1)$ -th Chow group  $A_{n-1}(X)$  is the group of linearly inequivalent divisors. We see the correspondence between one-dimensional cones, i.e. vectors, and divisors: A divisor is the toric variety constructed from a star of an one-dimensional cone. In the case of three dimensional resolved orbifolds the first Chow group  $A_1(X)$  is the group of complex curves: A complex curve in  $X$  can be described by the star of a two-dimensional cone. This means that in the case of interest, i.e. three dimensional non-compact resolved orbifolds, a point in the projected fan corresponds to a divisor and a line corresponds to a complex curve.

Let  $X$  be one of the resolved three-dimensional non-compact orbifolds. We can now show that if a vector, i.e. a point in the projected fan, is inside the triangle formed by  $v_1, v_2$ , and  $v_3$ , then the corresponding divisor is compact. Let us assume that the inner vector  $t$  has the components  $(t_1, t_2, 1)$ . Let  $\sigma$  denote the one-dimensional cone spanned by  $t$ . To obtain  $\text{Star}(\sigma)$  we do the following: we take  $t$  as a basis vector instead of  $(0, 0, 1)$  and compute the components of other vectors involved, i.e. those which span the cones which have  $\sigma$  as their face. Then, because  $t$  is now the third basis vector, the projection corresponds to just dropping the third component. This procedure can very intuitively understood from the projected fans. Figure A.1 shows an example. We just have to take the point corresponding to  $t$  in the projected fan as the origin and the lines going out from it as the vectors spanning one-dimensional cones in the star. Even if there were no other points than  $v_1, v_2, v_3$ , and  $t$  in the projected fan, the resulting toric variety  $V(\sigma)$  will be compact since  $\text{Star}(\sigma)$  contains three two-dimensional cones whose union is the whole two-dimensional plane, i.e.,

$$|\text{Star}(\sigma)| = \mathbb{R}^2.$$

If there are more points in the projected fan, there will be more lines, i.e. the two-dimensional cones in  $\text{Star}(\sigma)$  will become “finer”, but their union will still be the whole plane. Therefore  $V(\sigma)$  is compact. Using the same arguments, it is obvious that outer points in the projected fan yield non-compact divisors.

The case of compact curves is treated analogously to divisors. A line in the projected fan means a two-dimensional cone  $\tau$  in the fan. Let us denote the vectors spanning  $\tau$  by  $t_1$  and  $t_2$ . In the fully triangulated fan,  $\tau$  is a shared face of two basic cones. Figure A.2 illustrates the situation. This means



**Figure A.3:** Example of the star of a two-dimensional cone  $\tau$

that we only have to consider four vectors altogether, namely  $t_1, t_2$ , and  $u_1, u_2$ . The vectors  $u_1$  and  $u_2$  span a three dimensional cones  $\sigma_1$  and  $\sigma_2$  respectively with  $t_1$  and  $t_2$ . Note that  $(u_1, t_1, t_2)$  and  $(u_2, t_1, t_2)$  form each a  $\mathbb{Z}$ -basis of  $N = \mathbb{Z}^3$ . Moreover, since  $u_2$  is relative to  $u_1$  on the “opposite side” of the hyperplane spanned by  $t_1$  and  $t_2$ ,

$$1 = \det(u_1, t_1, t_2) = -\det(u_2, t_1, t_2) \quad (\text{A.15})$$

assuming  $u_1$  is positively oriented. Consider now the equation

$$\begin{pmatrix} u_{21} & t_{11} & t_{21} \\ u_{22} & t_{12} & t_{22} \\ u_{23} & t_{13} & t_{23} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} \quad \text{with } a, b, c \in \mathbb{Z}.$$

This equation has an unique solution because of (A.15). Thus

$$u_1 = au_2 + N(\tau).$$

In  $\text{Star}(\tau)$  the vector  $u_1$  is a  $\mathbb{Z}$ -multiple of  $u_2$ . The coefficient  $a$  is negative since the orientation of  $u_1$  and  $u_2$  is opposite relative to the hyperplane spanned by  $t_1$  and  $t_2$ . Setting  $p(u_2) = (1) \in N(\tau)$ , we obtain  $p(u_1) = (a)$ . Consequently,

$$|\text{Star}(\tau)| = \mathbb{R}$$

and the resulting toric variety  $V(\tau)$ , which is a complex curve, is compact. The coefficient  $a$  is equal to  $-1$  since the curve has to be compact and smooth. The only one-dimensional compact and smooth toric variety is  $\mathbb{P}^1$  and its fan consists of  $(1)$  and  $(-1)$ . The fact that  $a = -1$  we can also see from the projected fans of the resolved orbifolds. Using the same arguments, we can easily see that outer lines correspond to non-compact curves.

Let us look at an example. Figure A.3 shows the triangulation 6.3(a) of  $\text{Res}(\mathbb{C}^3/\mathbb{Z}_{6-II})$ . Note that we renamed the vectors to be consistent with the discussion above. We want to construct the star of  $\tau$  which is spanned by  $t_1$  and  $t_2$ . The vectors in components are

$$u_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We have

$$u_1 = -u_2 + 3t_1 - t_2 = -u_2 + N(\tau).$$

Consequently,  $\text{Star}(\tau)$  consists of  $(-1) = p(u_1)$  and  $(1) = p(u_2)$  which is the fan of  $\mathbb{P}^1$ .

To summarize, inner points and inner lines in the projected fans of resolved three-dimensional orbifolds correspond to compact divisors and compact curves respectively.

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